

# Steiner trees for fixed orientation metrics

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**Abstract** We consider the problem of constructing Steiner minimum trees for a metric defined by a polygonal unit circle (corresponding to  $\sigma \geq 2$  weighted legal orientations in the plane). A linear-time algorithm to enumerate all angle configurations for degree three Steiner points is given. We provide a simple proof that the angle configuration for a Steiner point extends to all Steiner points in a full Steiner minimum tree, such that at most six orientations suffice for edges in a full Steiner minimum tree. We show that the concept of canonical forms originally introduced for the uniform orientation metric generalises to the fixed orientation metric. Finally, we give an  $O(\sigma n)$  time algorithm to compute a Steiner minimum tree for a given full Steiner topology with  $n$  terminal leaves.

**Keywords** Steiner tree problem · Normed plane · Fixed orientation metric · Canonical form · Fixed topology

## 1 Introduction

Given a fixed set of *legal* orientations in the plane, we consider the problem of constructing minimum-length interconnection networks with the restriction that line segments use legal orientations only. Furthermore, we assign *weights* to the legal orientations, such that the cost of a line segment is its length times its weight, and the total cost of the network is the sum of weighted line segment costs. This problem is equivalent to the *Steiner tree problem* under a metric that has a *polygonally* bounded unit circle.

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The Steiner tree problem for fixed orientation metrics has important applications in VLSI design, where millions of nets should be routed on a (small) number of chip layers. Each layer is typically assigned a fixed orientation in order to make joint routing of multiple nets feasible. Traditionally only horizontal and vertical routing orientations have been used in chip production, but recent advances in production and routing algorithms have made it feasible to perform routing in more than two perpendicular orientations. The best known examples of such recent routing paradigms are the Y-architecture [9, 10] (where three uniformly distributed orientations are used) and the X-architecture [30, 32] (where four uniformly distributed orientations are used). Furthermore, there is a growing need for assigning weights to individual layers, such that some orientations can be preferred or deferred [34].

In addition to applications in VLSI design, fixed orientation metrics can be used to *approximate* arbitrary metrics in the plane. As shown in this paper, increasing the number of legal orientations only increases the running times of computing Steiner minimum trees (SMTs) linearly. Therefore it is feasible to make tight approximations of arbitrary unit circles by polygonal unit circles.

Algorithms for solving several fixed orientation distance problems were proposed by Widmayer et al. [31]. Lin and Xue [23] gave an efficient algorithm for embedding a minimum spanning tree under a fixed orientation metric.

For the Steiner tree problem, the rectilinear case (where two perpendicular orientations are given) has been studied most intensively due to its applications in traditional VLSI design [14–16, 36]. For the uniform orientation metric—where the given orientations are uniformly distributed (are separated by a fixed angle)—important algorithms and fundamental properties for SMTs were given in [1–3, 21, 24, 28]. Heuristics and efficient algorithms for special cases of the problem in uniform orientation metrics were considered in [22, 25]. Li et al. [20] considered the general (non-weighted) Steiner tree problem in fixed orientation metrics, and gave some results related to the so-called generalized Hanan grid.

The Steiner tree problem in fixed orientation metrics is NP-hard, but when the *topology* of the tree is known, the problem can be solved in polynomial time by using linear programming [33, 37]. Moreover, for the uniform orientation problem with  $\lambda \geq 2$  orientations—and assuming that the topology is *full* (i.e., non-degenerate) and has  $n$  terminal points as leaves—an  $O(\lambda n)$  algorithm was proposed by Brazil et al. [5]. (Note that this algorithm is linear in  $n$  for constant  $\lambda$ .) The concept of forbidden subpaths from [3, 4] played a crucial role in achieving this fast algorithm.

In this paper we generalize the linear-time algorithm in [5] to arbitrary fixed orientation problems. We show that the result can be achieved without using the concept of forbidden subpaths. Our results also nicely complement similar results, such as those of Du et al. [12], for Steiner trees in any normed space defined by a strictly convex and differentiable unit circle.

We have endeavoured, in this paper, to find simple, direct proofs of these results. Particularly worthy of mention is Theorem 5.2, which shows that each full and fulsome component of an SMT only requires a small number of different orientations. Not only is this a key to establishing the canonical forms in the later parts of the paper, but it also provides a much simpler alternative proof to the less general and highly technical results on directions and forbidden subpaths in SMTs for uniform orientation metrics in [3–5]. Our paper essentially subsumes the main results of these papers, and many other structural results that have appeared in the literature on uniform orientation (including rectilinear) SMTs.

We begin by generalising the concept of direction sets from [5] to fixed orientation metrics and present a  $\Theta(\sigma)$  time algorithm to enumerate all direction sets for an arbitrary fixed orientation metric with  $\sigma \geq 2$  legal orientations. Then we show that direction sets extend

throughout full SMTs—meaning that the edges in a full SMT use at most six legal orientations. In addition, we show that these full SMTs can be assumed to have certain canonical forms; one of these canonical forms has the property that the edges of a full SMT use at most four legal orientations. Furthermore, on the basis of the so-called depth-first order canonical form we give an  $O(\sigma n)$  time algorithm to determine an SMT for a given full (i.e., non-degenerate) Steiner topology with  $n$  terminal leaves.

The paper is organised as follows. In Sect. 2 we define the Steiner tree problem in normed planes formally, present some known results and fix our notation. In Sect. 3 we present a useful geometric characterisation of meeting angles for Steiner points, and in Sect. 4 we introduce the concept of direction sets and show that we can identify all direction sets in  $\Theta(\sigma)$  time. We show that a full SMT uses at most six legal orientations (coming from a single direction set) in Sect. 5. In Sect. 6 we give some fundamental results on length-preserving perturbations of Steiner points, and we introduce the concept of canonical forms for full SMTs. Based on these canonical forms, in Sect. 7 we devise a linear-time algorithm to compute an SMT for a given full Steiner topology. We conclude the paper in Sect. 8.<sup>1</sup>

## 2 Preliminaries

We begin by setting some basic definitions and notation, and establishing generalisations of some important properties of SMTs for uniform orientation metrics. For a more detailed development of these properties, see [2, 5].

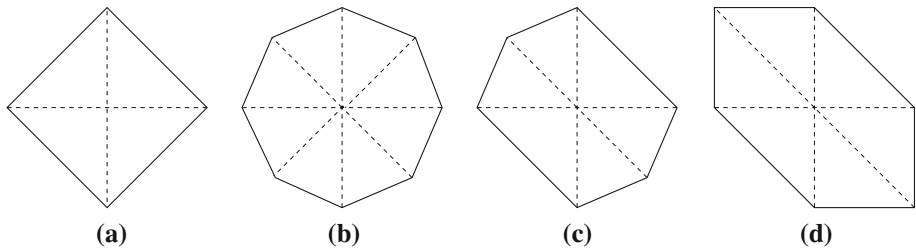
### 2.1 Normed planes

Given a compact, convex, centrally symmetric domain  $\mathcal{D}$  in the Euclidean plane  $E^2$ , one can define a norm  $\|\cdot\|_{\mathcal{D}} : E^2 \rightarrow \mathbf{R}^+$  by setting  $\|\mathbf{x}\|_{\mathcal{D}} = r$  where  $\mathbf{x} = r\mathbf{u}$  and  $\mathbf{u} \in \partial\mathcal{D}$ , the boundary of  $\mathcal{D}$ . Denote the (closed) line segment joining distinct points  $\mathbf{x}$  and  $\mathbf{y}$  by  $\mathbf{xy}$ . We can then define a metric  $|\cdot|_{\mathcal{D}}$  on  $E^2$  by setting  $|\mathbf{xy}|_{\mathcal{D}} = \|\mathbf{x} - \mathbf{y}\|_{\mathcal{D}}$ . The resulting metric space  $M = (E^2, |\cdot|_{\mathcal{D}})$  is called a *normed* (or *Minkowski*) plane with unit disc  $\mathcal{D}$ . By definition,  $\partial\mathcal{D} = \{\mathbf{x} : \|\mathbf{x}\|_{\mathcal{D}} = 1\}$ , and is often referred to as the unit circle.

In this paper we will limit our attention to normed planes in which the unit circle  $\partial\mathcal{D}$  is a centrally symmetric polygon. The best known example of this is the rectilinear plane [14] with norm  $\|(a, b)\|_{\mathcal{D}} = |a| + |b|$ . Here  $\partial\mathcal{D}$  is a square whose two diagonals lie on the  $x$ -axis and  $y$ -axis, respectively (Fig. 1a). An important generalisation of the rectilinear plane is the  $\lambda$ -geometry plane [2, 28], for each integer  $\lambda (\geq 2)$ , in which  $\partial\mathcal{D}$  is a regular  $2\lambda$ -gon (Fig. 1b). Further generalisation is denoted fixed orientation geometry [31] (Fig. 1c).

In the  $\lambda$ -geometry plane, or in any normed plane in which  $\partial\mathcal{D}$  is a centrally symmetric  $2\sigma$  polygon inscribed in a (Euclidean) unit circle, we can avoid the difficulties of computing directly with the associated metric as follows. We assume that line segments are restricted to lying in  $\sigma$  distinct *legal orientations* corresponding to the  $\sigma$  diagonals of  $\partial\mathcal{D}$ . Then, for any pair of distinct points  $\mathbf{x}$  and  $\mathbf{y}$ , the length of the shortest path between them composed of legal line segments is equal to  $|\mathbf{xy}|_{\mathcal{D}}$ . In other words, line segments in the normed plane can be represented by minimum paths composed of legal line segments in the Euclidean plane. See [31] for more details on the properties of such a representation. This will be our point of view throughout this paper.

<sup>1</sup> Due to space restrictions, a few of the more straightforward proofs in this paper have been omitted. An expanded version of this paper, including all proofs and some additional minor results, can be obtained on-line [7].



**Fig. 1** Examples of polygonal unit circles. (a) Rectilinear (2 perpendicular orientations). (b) Uniform orientations ( $\lambda = 4$ ). (c) Non-uniform orientations ( $\sigma = 3$ ). (d) General weighted non-uniform orientations

As noted by Swanepoel [29], this idea can be extended to normed planes in which  $\partial\mathcal{D}$  is any centrally symmetric  $2\sigma$  polygon—not necessarily inscribed in a Euclidean unit circle (Fig. 1d). Again line segments in the normed plane can be represented by paths composed of legal line segments, but now the total length of such a path is determined by weighting each legal line segment according to its orientation. The weighting on each line segment is the inverse of the distance of the corresponding vertex of  $\partial\mathcal{D}$  from the centre of  $\mathcal{D}$ . One can again show, using similar arguments to those in [31], that the minimum length of such a path corresponds to the normed plane distance metric between its endpoints.

It is natural to ask whether this approach can be extended to any given set of  $\sigma$  orientations with given (positive) weights for each orientation. A potential problem is that the corresponding  $2\sigma$  polygon may not be convex, meaning that we do not have a normed plane. However, we can circumvent this difficulty by considering the convex hull of the corresponding polygonal region. For any vertex of the  $2\sigma$  polygon that does not lie on this convex hull, the corresponding orientation will never appear in a minimum path because a line segment in that orientation can always be replaced by line segments in other legal orientations at a smaller total cost. Hence, in this sense our approach is completely general for all finite sets of orientations and all possible (positive) weightings.

## 2.2 Steiner trees in normed planes with polygonally bounded unit circle

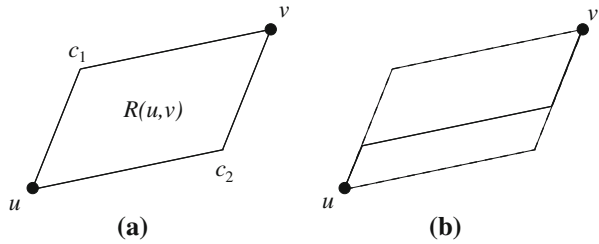
Throughout this paper, we assume that a normed plane whose unit circle  $\partial\mathcal{D}$  is a centrally symmetric polygon is given. Let  $\sigma \geq 2$  denote the number of distinct legal orientations in the normed plane.

Given a set of points  $N$  (so-called *terminals*) in the plane, a *Steiner minimum tree* (SMT) is a shortest possible interconnection of  $N$  under the metric given by  $\partial\mathcal{D}$ . An SMT may contain vertices of degree three or more that are not among the given terminals; these vertices are denoted *Steiner points*.

The graph structure of a tree  $T$  (i.e., the pattern of adjacencies of the vertices for a given labelling of the terminals) is referred to as its *topology*  $\mathcal{T}$ . If, for the given norm, the total edge length of  $T$ ,  $|T|_{\mathcal{D}}$ , is minimum for a given topology  $\mathcal{T}$  (possibly in a degenerate way), then we say that  $T$  is an SMT *for*  $\mathcal{T}$ . For a fixed topology it follows that  $|T|_{\mathcal{D}}$ , treated as a function of the Steiner points of  $T$ , is a convex function (since  $T$  lies in a normed plane). Note, however, that this convexity is not strict; it may be possible to move Steiner points in an SMT without increasing the length of the tree.

A tree  $T$  or its topology  $\mathcal{T}$  is said to be *full* if all its terminals have degree 1; if every Steiner point furthermore has degree three, then  $\mathcal{T}$  is denoted a *full Steiner topology*. Given any SMT  $T$ , we can decompose  $T$  into full components, that is, into full SMTs that meet

**Fig. 2** Embedding of a bent edge. (a) The two embeddings  $uc_1v$  and  $uc_2v$  that only require two line segments (and a single corner point); the union of all shortest paths between  $u$  and  $v$  is denoted by  $R(u, v)$ . (b) An embedding with identical orientations at endpoints



only at terminals. Furthermore,  $T$  or its topology  $\mathcal{T}$  is said to be *fulsome* if  $T$  contains the maximum possible number of full components for any SMT on the terminal set of  $T$ . If we perturb the Steiner points in a fulsome SMT (without changing the length of the tree), then we cannot make a Steiner point coincide in position with one of the terminals. Clearly, for any set of terminals there always exists an SMT in which every full component is fulsome.

The edges of Steiner trees will be assumed to be represented by paths composed of line segments in legal orientations, as discussed in Sect. 2.1. It follows from [31] that each edge  $(u, v)$  of a Steiner tree uses at most two legal orientations; furthermore these two orientations are necessarily adjacent. When two legal orientations are required to represent an edge, then we call the edge a *bent edge*. Bent edges can be realised using exactly two legal line segments. There are two possible embeddings of such a path in the plane  $E^2$ , as shown in Fig. 2a. A point (of degree 2) at which two legal line segments with different orientations meet is called a *corner point*. There are, of course, many other possible embeddings using multiple corner points, and the union of all these embeddings (or shortest paths) is a parallelogram  $R(u, v)$ . It is sometimes useful to consider an embedding with two corner points (as shown in Fig. 2b) which has the property that the path has the same orientation at both endpoints.

### 3 Meeting angles for Steiner points

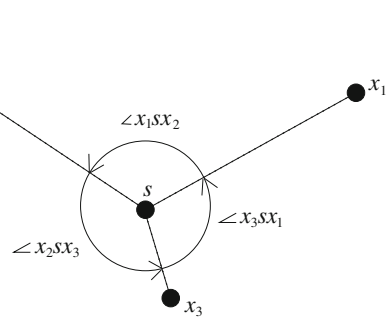
In this section we give a complete characterisation of the distribution of meeting angles of Steiner points in SMTs for a metric defined by unit circle  $\partial\mathcal{D}$ . Let  $\mathbf{u}_l, l = 0, \dots, 2\sigma - 1$  be the  $2\sigma$  vectors that define the vertices of the unit circle  $\partial\mathcal{D}$  (in counter-clockwise order around the circle). These are unit vectors in the metric given by  $\partial\mathcal{D}$ , that is, they have length 1 under metric  $|\cdot|_{\mathcal{D}}$ . By the central symmetry of  $\partial\mathcal{D}$ , we have that  $\mathbf{u}_l = -\mathbf{u}_{(l+\sigma) \bmod 2\sigma}$ . Each unit vector corresponds to a *legal direction*, that is, under this metric, any (oriented) legal line segment must use one of the  $2\sigma$  unit vector directions. The *successor* of unit vector  $\mathbf{u}_l$  is the vector  $\mathbf{u}_{l+1}$  where  $l + 1 := 0$  if  $l = 2\sigma - 1$ .

In the remainder of this paper we will make frequent use of the assumption that  $\partial\mathcal{D}$  is placed somewhere in the plane such that its centre coincides with a given point  $s$ . For such a fixed centre we can unambiguously refer to the *vertices* of  $\partial\mathcal{D}$ , which are the endpoints of the unit vectors  $\mathbf{u}_l$  for this fixed centre. The vertex that corresponds to the endpoint of unit vector  $\mathbf{u}_l$  is denoted by  $u_l$  for  $l = 0, \dots, 2\sigma - 1$ .

#### 3.1 Steiner configurations

Following [29], we define a *Steiner configuration* in the normed plane as a star with centre  $s$  and leaves  $x_1, \dots, x_m$  (with  $s, x_1, \dots, x_m$  all distinct) that is part of some SMT with Steiner point  $s$ . By [29], Steiner points in a normed plane have degree at most 4, hence  $m = 3$  or

**Fig. 3** Meeting angles



4 in a Steiner configuration. Here we first consider the case where  $m = 3$ , and return to the (special)  $m = 4$  case in Sect. 5.

Consider a star with Steiner point  $s$  and leaves  $x_1, x_2$  and  $x_3$ . Assume w.l.o.g. that the counter-clockwise order of the leaves is  $x_1, x_2, x_3$ . We define the *meeting angles* of  $s$  to be the angles  $\angle x_1sx_2, \angle x_2sx_3$  and  $\angle x_3sx_1$ , i.e., the angles that appear counter-clockwise around the Steiner point  $s$  (Fig. 3).

Du et al. [12] have proved the following result for strictly convex and differentiable unit circles (which includes the Euclidean metric): If one of the edges of the star, say  $sx_1$ , is given, then the orientations of the other edges  $sx_2$  and  $sx_3$  are *unique*. In other words, if the Steiner point  $s$  and one of the leaves, say  $x_1$ , are given, then the meeting angles of  $s$  are unique. In this section we develop an analogous version of this result for polygonal unit circles. We start with the following theorem from [26], which holds for any normed plane.

**Definition 3.1** An angle  $\phi = \angle x_1sx_2$  is *critical* if  $\phi \leq \pi$  and there exists a point  $x_3 \neq s$  such that Steiner point  $s$  with leaves  $x_1, x_2, x_3$  forms a Steiner configuration, with  $\phi = \angle x_1sx_2$  as one of the meeting angles. (Compared to the definition of critical in [26] we have added the condition that the meeting angle is at most  $\pi$ .)

**Theorem 3.1** [26] *A star with centre  $s$  and leaves  $x_1, x_2, x_3$  forms a Steiner configuration in a normed plane if and only if all meeting angles are critical.*

Chakerian and Ghandehari [8] gave a very useful characterisation of Steiner configurations for strictly convex and differentiable unit circles. If a unit circle is placed with its centre at  $s$  and  $l_1, l_2$  and  $l_3$  are the tangents of the unit circle where the rays  $s \rightarrow x_1, s \rightarrow x_2$  and  $s \rightarrow x_3$  intersect the unit circle, then  $l_1, l_2$  and  $l_3$  form a triangle whose *centroid* coincides with  $s$ . We say that the tangents  $l_1, l_2$  and  $l_3$  have the *centroid-property*. For more general unit circles, including polygonal unit circles, the tangents are not in general well-defined. However, by considering supporting lines instead of tangents, the following slightly weaker result can be proved for arbitrary normed planes (or unit circles):

**Lemma 3.1** [26] *The following statements are equivalent in a normed plane with unit circle  $\partial\mathcal{D}$ :*

- (i)  $\angle x_1sx_2$  is a critical angle
- (ii) if the unit circle  $\partial\mathcal{D}$  with centre  $s$  intersects the ray  $s \rightarrow x_1$  at  $x'_1$  and the ray  $s \rightarrow x_2$  at  $x'_2$ , then there exist lines  $l_1, l_2$  and  $l_3$  supporting  $\partial\mathcal{D}$ ,  $l_1$  at  $x'_1$  and  $l_2$  at  $x'_2$ , such that  $s$  is the centroid of the triangle formed by  $l_1, l_2$  and  $l_3$  (or equivalently,  $l_1, l_2$  and  $l_3$  have the centroid-property).

### 3.2 Steiner configurations for fixed orientation metrics

Based on Lemma 3.1, we give a characterisation of Steiner configurations for polygonal unit circles that is similar to the one shown by Chakerian and Ghandehari [8]. In order to do so, we need to characterise under which conditions a given supporting line  $l_1$  of the unit circle can be a side in a triangle of supporting lines that has  $s$  as its centroid. Our characterisation will be based on the following condition on when a set of supporting lines  $l_1, l_2$  and  $l_3$  has the centroid-property.

**Lemma 3.2** [12] *Let  $l_0$  be the line that is parallel to  $l_1$  and contains  $s$ . Let  $\Delta$  be the Euclidean distance between the parallel lines  $l_1$  and  $l_0$ . Consider the line  $L$  that is parallel to  $l_1$  at distance  $3\Delta$  from  $l_1$  (and at distance  $2\Delta$  from  $l_0$ ). Let  $w_2$  be the intersection of  $l_2$  with  $l_0$  and let  $w_3$  be the intersection of  $l_3$  with  $l_0$  (Fig. 4a). Then supporting lines  $l_1, l_2$  and  $l_3$  satisfy the centroid property if and only if (i) lines  $l_2$  and  $l_3$  intersect somewhere on the line  $L$ , and (ii)  $|sw_2| = |sw_3|$ , where  $|\cdot|$  denotes the Euclidean distance.*

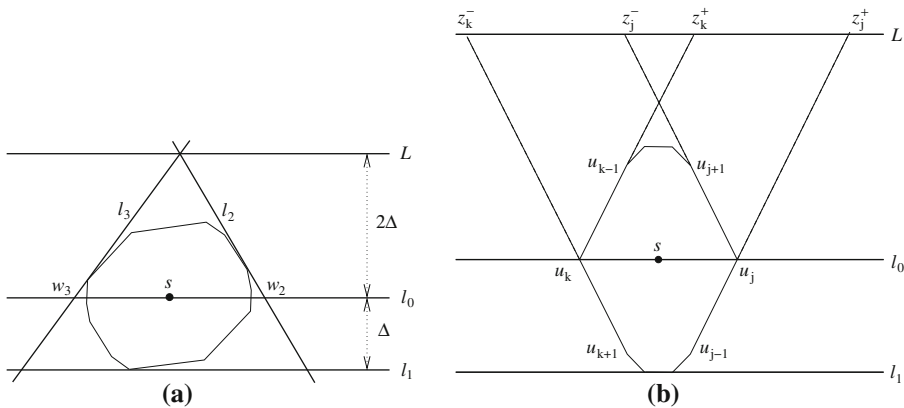
In the following lemmas and theorems we will frequently refer to  $l_0$  and  $L$  as defined by Lemma 3.2. First we give two simple lemmas that further characterize sets of supporting lines that fulfill the centroid-property (proofs are given in [7]).

**Lemma 3.3** *Let  $l_1, l_2$  and  $l_3$  be a set of supporting lines that fulfils the centroid-property, and let  $l_0$  be the line that is parallel to  $l_1$  and contains  $s$ . Then neither  $l_2$  nor  $l_3$  can support  $\partial\mathcal{D}$  at a point that is strictly between  $l_1$  and  $l_0$ .*

**Lemma 3.4** *Let  $l_1$  be a line that supports a unit circle  $\partial\mathcal{D}$  with centre  $s$ , and let  $l_0$  be the line that is parallel to  $l_1$  and contains  $s$ . If  $l_0$  does not intersect a vertex of  $\partial\mathcal{D}$  then there exists exactly one pair of supporting lines  $l_2$  and  $l_3$ , such that  $l_1, l_2$  and  $l_3$  have the centroid-property.*

If, on the other hand,  $l_0$  does intersect a vertex of  $\partial\mathcal{D}$  then by the central symmetry of  $\partial\mathcal{D}$  it intersects two vertices. We consider this case in the following lemma.

**Lemma 3.5** *Let  $l_1$  be a line that supports a unit circle  $\partial\mathcal{D}$  with centre  $s$ , and let  $l_0$  be the line that is parallel to  $l_1$  and contains  $s$ . If  $l_0$  intersects  $\partial\mathcal{D}$  at two vertices  $u_j$  and  $u_k$  then either*



**Fig. 4** Centroid-property. (a) Geometric characterisation of triangle with centroid  $s$ . (b) Illustration of proof of Lemma 3.5

- there exists exactly one pair of supporting lines  $l_2$  and  $l_3$ , such that  $l_1, l_2$  and  $l_3$  have the centroid-property, or
- there exists an infinite set of supporting lines  $l_2$  and  $l_3$  such that  $l_1, l_2$  and  $l_3$  have the centroid-property, and in each case  $l_2$  and  $l_3$  support  $\partial\mathcal{D}$  at  $u_j$  and  $u_k$ , respectively.

*Proof* Recall that  $u_j$  and  $u_k$  are the vertices of  $\partial\mathcal{D}$  that are intersected by  $l_0$ . The predecessor of  $u_j$  (resp.  $u_k$ ) on  $\partial\mathcal{D}$  is  $u_{j-1}$  (resp.  $u_{k-1}$ ) and the successor is  $u_{j+1}$  (resp.  $u_{k+1}$ ).

Let  $z_j^-$  be the intersection of the line through  $u_j$  and  $u_{j+1}$  with  $L$ , and let  $z_j^+$  be the intersection of the line through  $u_{j-1}$  and  $u_j$  with  $L$ ; define  $z_k^-$  and  $z_k^+$  similarly (Fig. 4b).

Now we distinguish between three cases, depending on the order (from left to right) in which the points  $z_j^-$  and  $z_k^+$  appear on  $L$ :

1.  $z_j^- < z_k^+$ : In this case any point  $z$  in the interval  $[z_j^-, z_k^+]$  defines a pair of supporting lines  $l_2$  and  $l_3$  that contain  $z$  and support  $\partial\mathcal{D}$  on opposite sides, such that  $l_1, l_2$  and  $l_3$  have the centroid-property. In this case we have an infinite number of supporting line pairs  $l_2$  and  $l_3$ , and all these pairs support  $\partial\mathcal{D}$  at  $u_j$  and  $u_k$ , respectively.
2.  $z_j^- = z_k^+$ : Here the point  $z = z_j^- = z_k^+$  defines a pair of supporting lines  $l_2$  and  $l_3$  that contain  $z$  and support  $\partial\mathcal{D}$  on opposite sides, such that  $l_1, l_2$  and  $l_3$  have the centroid-property. Note that  $l_2$  supports  $\partial\mathcal{D}$  both at  $u_j$  and  $u_{j+1}$  (is a tangent), and similarly  $l_3$  supports  $\partial\mathcal{D}$  both at  $u_k$  and  $u_{k-1}$ . In this case this is the only pair of supporting lines that jointly with  $l_1$  have the centroid-property.
3.  $z_j^- > z_k^+$ : In this case there is clearly no pair of supporting lines that (jointly with  $l_1$ ) have the centroid-property and support  $\partial\mathcal{D}$  at either  $u_j$  or  $u_k$ . It follows that the method of proof of Lemma 3.4 applies to this case; hence, exactly one pair of supporting lines jointly with  $l_1$  has the centroid-property. The common intersection point for this pair of supporting lines lies strictly between  $z_k^+$  and  $z_j^-$  on  $L$ . □

We are now ready to prove the main result of this section—namely to make a precise link between the centroid-property and Steiner configurations for polygonal unit circles.

**Theorem 3.2** *Consider a normed plane defined by a polygonal unit circle  $\partial\mathcal{D}$ , and a star  $\mathcal{S}$  with centre  $s$  and leaves  $x_1, x_2$  and  $x_3$ . In the unit circle with centre  $s$ , let  $x'_1, x'_2$  and  $x'_3$  be the intersections with (the extensions of) the edges  $sx_1, sx_2$  and  $sx_3$ , respectively. Assume that one of the intersections, say  $x'_1$ , is not a vertex of the polygonal unit circle. Then  $\mathcal{S}$  forms a Steiner configuration if and only if there exists a set of lines  $l_1, l_2$  and  $l_3$  supporting  $\partial\mathcal{D}$  at  $x'_1, x'_2$  and  $x'_3$ , respectively, such that  $l_1, l_2$  and  $l_3$  have the centroid-property.*

*Proof* Consider first the case where we have a set of supporting lines  $l_1, l_2$  and  $l_3$  that satisfy the centroid-property. By Lemma 3.1, each of the meeting angles in the star  $\mathcal{S}$  are therefore critical angles, since we can use supporting lines  $l_1, l_2$  and  $l_3$  to prove this for all three meeting angles. Hence, by Theorem 3.1, the star  $\mathcal{S}$  forms a Steiner configuration.

For the other direction, assume that  $\mathcal{S}$  forms a Steiner configuration. All meeting angles of  $\mathcal{S}$  are therefore critical by Theorem 3.1. For each meeting angle in  $\mathcal{S}$ , we know by Lemma 3.1 that there exist supporting lines having the centroid-property. However, these sets of supporting lines need not be identical: the supporting lines for, e.g., meeting angle  $\angle x_1 s x_2$  support  $x'_1$  and  $x'_2$  on  $\partial\mathcal{D}$ , but need not support the third point  $x'_3$  on  $\partial\mathcal{D}$ . We will now prove that there indeed exists a set of supporting lines with the centroid-property supporting *all* three points  $x'_1, x'_2$  and  $x'_3$ —thus finishing the proof of the theorem.

By the assumptions of the theorem we know that a supporting line of  $x'_1$  is necessarily a unique tangent; let  $l_1$  denote this tangent. From Lemmas 3.4 and 3.5 we know that  $l_1$  is either



part of a *unique* set of supporting lines with the centroid-property *or* is part of an infinite set of such supporting lines. If  $l_1$  is part of a *unique* set of supporting lines  $l_1, l_2$  and  $l_3$ , then this set is the only one that proves that meeting angles  $\angle x_1 s x_2$  and  $\angle x_3 s x_1$  are critical. This set of supporting lines must therefore support  $\partial\mathcal{D}$  in  $x'_1, x'_2$  and  $x'_3$  simultaneously.

Now assume that there exists an infinite set of supporting lines  $l_1, l_2$  and  $l_3$  that have the centroid-property. By Lemma 3.5, we have the situation illustrated in Fig. 4b. Each possible supporting line  $l_2$  supports some fixed vertex  $u_j$  of  $\partial\mathcal{D}$ , and each possible supporting line  $l_3$  supports another fixed vertex  $u_k$  of  $\partial\mathcal{D}$ . As a consequence,  $x'_2$  must be somewhere on the line segment  $S_j$  of  $\partial\mathcal{D}$  defined by vertices  $u_j$  and  $u_{j+1}$ ; similarly,  $x'_3$  must be somewhere on the line segment  $S_k$  defined by vertices  $u_{k-1}$  and  $u_k$ . There are three cases to consider:

- $x'_2 = u_j$ : In this case we can pick  $l_3$  such that it overlaps with  $S_k$ , and choose  $l_2$  such that  $l_1, l_2$ , and  $l_3$  have the centroid-property. Since  $l_3$  overlaps with  $S_k$ , it supports any possible point  $x'_3$  on  $S_k$ .
- $x'_3 = u_k$ : This case is symmetric to the above—here we let  $l_2$  overlap with  $S_j$  and choose  $l_3$  accordingly.
- $x'_2 \neq u_j$  and  $x'_3 \neq u_k$ : This case *cannot* occur, since the angle  $\angle x_2 s x_3$  is not critical. The reason is that the tangents  $l_2$  and  $l_3$  must overlap with  $S_j$  and  $S_k$ , respectively. However, there exists no third supporting line  $l_1$  such that  $l_1, l_2$  and  $l_3$  have the centroid-property. The intersection point between  $l_2$  and  $l_3$  is strictly between the lines  $l_0$  and  $L$  (in Fig. 4b), hence the third edge of the triangle must intersect the interior of the unit circle.  $\square$

The theorem requires that one of the intersections with the unit circle appears at a *tangent* point. However, this is no restriction as will be clear from the discussion in next section (in particular Lemma 4.1).

### 4 Direction sets

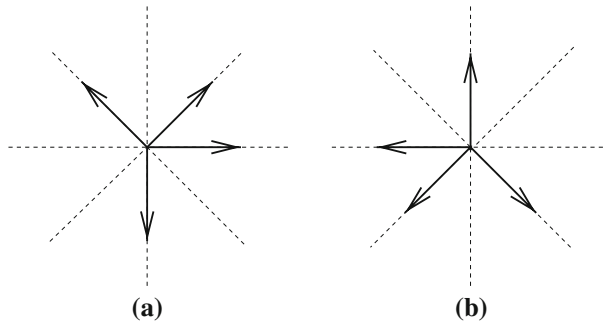
For any Steiner configuration of degree  $m = 3$  there is an associated set of legal directions, namely the legal directions used by all edges in the star (where directions are considered as oriented outward from the centre). A Steiner configuration  $S$  is said to be *maximal* if there exists no other Steiner configuration that uses a strict superset of the legal directions used by  $S$ . We define a *direction set* to be the set of legal directions used by a maximal Steiner configuration. Note that in  $\lambda$ -geometry these direction sets are the same as the ‘feasible direction sets’ defined in [5].

For a direction set  $\mathcal{U}$  we define the *complementary* direction set as the direction set that is obtained by reversing all directions in  $\mathcal{U}$  (Fig. 5). By the central symmetry of  $\partial\mathcal{D}$ , direction sets appear as *pairs* of complementary direction sets.

**Lemma 4.1** *A direction set contains at least 4 and at most 6 distinct directions.*

*Proof* The upper bound is obvious if  $m = 3$ , since each edge uses at most two legal directions. We establish the lower bound using a continuity argument.

Suppose, contrary to the claim of the lemma, there exists a direction set  $\mathcal{U}$  with only three directions. Let  $T$  be an SMT with three terminals,  $t_1, t_2, t_3$  and a Steiner point  $s$ , such that  $T$  has direction set  $\mathcal{U}$ . By convexity, the choice of  $s$  is unique, since if there were a second Steiner point  $s'$  that gave an SMT then every point in the line segment  $ss'$  would also be the Steiner point of an SMT. This would mean that a larger direction set strictly containing  $\mathcal{U}$  could be found by moving the Steiner point from  $s$  a small distance towards  $s'$ .



**Fig. 5** Complementary direction sets in  $\lambda$ -geometry for  $\lambda = 4$

For all points  $p$  in the plane define the function  $f(p) := |t_1 p|_{\mathcal{D}} + |t_2 p|_{\mathcal{D}} + |t_3 p|_{\mathcal{D}}$ . Since  $T$  is an SMT it follows that, for every  $p$ ,  $f(p) \geq f(s) = |T|_{\mathcal{D}}$ . For every  $\varepsilon > 0$ , let  $B_\varepsilon$  be the open Euclidean ball with centre  $s$  and radius  $\varepsilon$ . By the continuity of  $f$  it follows that for each  $\varepsilon$  there exists  $\delta = \delta(\varepsilon) > 0$  such that  $f(p) > f(s) + \delta$  for all  $p \notin B_\varepsilon$ . Clearly we can also assume that  $\delta < \varepsilon$ .

Now choose  $\varepsilon > 0$  sufficiently small such that perturbing one or both endpoints of each edge  $t_i s$  by at most  $\varepsilon$  results in an edge in the normed plane that still uses a direction from  $\mathcal{U}$  (for  $i \in \{1, 2, 3\}$ ). Suppose we move terminal  $t_1$  to  $t'_1$  not on the line through  $t_1 s$  such that  $0 < |t_1 t'_1|_{\mathcal{D}} < \delta(\varepsilon)$  and such that  $T'$ , the SMT for  $t'_1, t_2, t_3$ , satisfies  $|T'|_{\mathcal{D}} \leq |T|_{\mathcal{D}}$ . Let  $s'$  be a Steiner point for  $T'$ . Then

$$\begin{aligned} f(s') &\leq |T'|_{\mathcal{D}} + |t_1 t'_1|_{\mathcal{D}} \\ &< |T|_{\mathcal{D}} + \delta. \end{aligned}$$

Hence  $s'$  lies in  $B_\varepsilon$ . By our choice of  $\varepsilon$  it follows that  $\mathcal{U}'$ , the direction set of  $T'$ , contains  $\mathcal{U}$  as a subset (since the endpoints of the three edges of  $T$  have each been perturbed by at most  $\varepsilon$ ). Since exactly one of the terminals of  $T$  has been perturbed it is easy to see that the direction set of  $T'$  is not the same as the direction set for  $T$ . Hence  $\mathcal{U}$  is a *strict* subset of  $\mathcal{U}'$ , contradicting the statement that  $\mathcal{U}$  is a direction set.  $\square$

A maximal Steiner configuration always has at least one edge that contributes two legal directions to the corresponding direction set (by Lemma 4.1). As in [5] we colour one such edge red and the other edges green and blue, respectively, in counter-clockwise order from the red edge. We extend these colour labels to the directions in the direction set. A direction set with coloured directions is denoted a *coloured* direction set. A coloured direction set always contains two red directions and these are adjacent legal directions. For a given colour, we therefore have either one or two legal directions in a coloured direction set. When we have two directions, these are labelled as the (exclusively) *primary* and the (exclusively) *secondary* direction, respectively, in counter-clockwise order around the unit circle. When we have a single direction for a given colour, this direction can be labelled either primary or secondary.

Assume we fix a pair of adjacent red directions, which correspond to two unit vectors  $\mathbf{u}_i$  and  $\mathbf{u}_{i+1}$  of  $\partial\mathcal{D}$ . How many coloured direction sets exist for this pair of red edges? Using Lemmas 3.4 and 3.5 we conclude that there exist either *one* or *two* direction sets for a fixed pair of red directions: For the case considered in Lemma 3.4 there is exactly *one* direction set which is given by the unit vectors whose endpoints are supported by the unique supporting

lines  $l_1, l_2$  and  $l_3$  having the centroid-property (where  $l_1$  supports  $\partial\mathcal{D}$  at the endpoints of  $\mathbf{u}_i$  and  $\mathbf{u}_{i+1}$ ). For the first subcase considered in Lemma 3.5 (Fig. 4b) there are exactly two direction sets—one corresponding to a pair of green directions  $\mathbf{u}_j$  and  $\mathbf{u}_{j+1}$ , and a single blue direction  $\mathbf{u}_k$ ; the other corresponding to a single green direction  $\mathbf{u}_j$  and a pair of blue directions  $\mathbf{u}_{k-1}$  and  $\mathbf{u}_k$ . The other positions of  $l_2$  and  $l_3$  do not give direction sets as in each case the corresponding set of directions is not maximal. For the remaining subcases of Lemma 3.5 there is exactly one direction set. We summarise:

**Theorem 4.1** *There are at least  $2\sigma$  and at most  $4\sigma$  coloured direction sets, where  $\sigma$  is the number of legal orientations defined by  $\partial\mathcal{D}$ . More precisely, there are  $k$  pairs of complementary coloured direction sets, where  $\sigma \leq k \leq 2\sigma$ .*

In the remainder of this section, we first develop an  $O(\sigma)$  time algorithm to determine all (coloured) direction sets for a fixed (red) edge, that is, assuming that the adjacent directions of a red edge are given. Then we use this algorithm to develop an optimal  $\Theta(\sigma)$  time algorithm for determining all coloured direction sets.

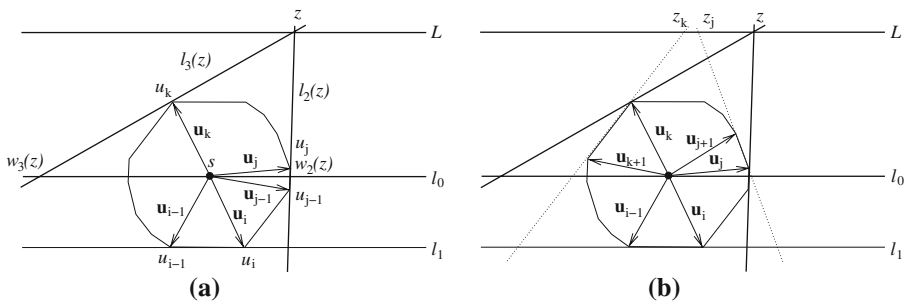
### 4.1 Linear-time algorithm for one fixed edge

In this section we give an  $O(\sigma)$  time algorithm to determine all coloured direction sets for a fixed pair of red directions. By the arguments of Theorem 4.1, there is either one direction set or two direction sets for a fixed pair of red directions. (By iterating over all  $2\sigma$  choices of adjacent red directions, we can identify all coloured direction sets in  $O(\sigma^2)$  time.)

Let  $\mathbf{u}_i$  and  $\mathbf{u}_{i+1}$  be a pair of red directions (here and in the following we identify directions with the unit vectors of  $\partial\mathcal{D}$ ). In order to determine the direction set(s) for this pair of red directions, we employ the construction used in the proof of Lemma 3.4.

Let  $l_1$  be the tangent supporting  $\partial\mathcal{D}$  at  $u_i$  and  $u_{i+1}$ , and define  $l_0$  and  $L$  as in Sect. 3.1. In counter-clockwise order around  $\partial\mathcal{D}$ , starting at  $u_{i+1}$ , let  $u_j$  be the first vertex that is on or above  $l_0$ . Define  $l_2(z)$  as the tangent supporting  $\partial\mathcal{D}$  at  $u_{j-1}$  and  $u_j$  (Fig. 6a), where  $z$  denotes the intersection of  $l_2(z)$  with  $L$ . Let  $l_3(z)$  be the other supporting line of  $\partial\mathcal{D}$  that intersects  $z$  (Fig. 6a). Line  $l_3(z)$  either supports  $\partial\mathcal{D}$  at a single vertex  $u_k$ , or at two adjacent vertices  $u_{k-1}$  and  $u_k$ . Clearly, given  $l_1$ , the supporting lines  $l_2(z)$  and  $l_3(z)$  can be determined in  $O(\sigma)$  time.

Now we simulate a continuous movement of the point  $z$  to the left along  $L$ . Let  $w_2(z)$  and  $w_3(z)$  be the intersections of  $l_2(z)$  and  $l_3(z)$ , respectively, with  $l_0$  (Fig. 6a). Note that initially we have  $|sw_2(z)| < |sw_3(z)|$ , where  $s$  is the centre of  $\partial\mathcal{D}$ . The movement of  $z$  is



**Fig. 6** Algorithm to determine direction set for fixed pair of red directions. (a) Initial position of  $z$ . (b) Computation of  $z_j$  and  $z_k$

continued until we have a point  $z^*$  such that  $|sw_2(z^*)| = |sw_3(z^*)|$ , that is, until we fulfill the centroid-property; by Lemmas 3.4 and 3.5 such a point must exist. After initialisation the algorithm is as follows:

1. Define  $z_j$  as the intersection of the line supporting  $\partial\mathcal{D}$  at  $u_j$  and  $u_{j+1}$  with  $L$ ; similarly, define  $z_k$  as the intersection of the line supporting  $\partial\mathcal{D}$  at  $u_k$  and  $u_{k+1}$  with  $L$  (Fig. 6b). Note that both  $z_j$  and  $z_k$  are strictly to the left of  $z$  on  $L$ ; let  $z'$  be the one of these two (possibly identical) points that is closest to  $z$ .
2. If  $|sw_2(z')| < |sw_3(z')|$ : If  $z_j = z'$  then set  $j = j + 1$ . If  $z_k = z'$  then set  $k = k + 1$ . Set  $z = z'$  and goto 1.
3. If  $|sw_2(z')| > |sw_3(z')|$ : The point  $z^*$  where  $|sw_2(z^*)| = |sw_3(z^*)|$  must lie strictly between  $z$  and  $z'$  on  $L$ . Hence there is a *unique* direction set where  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  are the two red directions:  $\mathbf{u}_j$  is the single green direction and  $\mathbf{u}_k$  the single blue direction (four directions in total).
4. If  $|sw_2(z')| = |sw_3(z')|$ :
  - (a)  $z_j = z'$  and  $z_k \neq z'$ : Here we have a *unique* direction set where  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  are the two red directions:  $\mathbf{u}_j$  and  $\mathbf{u}_{j+1}$  form a pair of green directions and  $\mathbf{u}_k$  the single blue direction (five directions in total).
  - (b)  $z_j \neq z'$  and  $z_k = z'$ : If either  $u_j$  or  $u_{k+1}$  are not on the line  $l_0$  through  $s$ , then we have a *unique* direction set where  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  are the two red directions:  $\mathbf{u}_k$  and  $\mathbf{u}_{k+1}$  form a pair of blue directions and  $\mathbf{u}_j$  a single green direction (five directions in total). If both  $u_j$  and  $u_{k+1}$  are on the line  $l_0$  through  $s$ , then we have *two* direction sets where  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  are the two red directions: The one just described, and one where  $\mathbf{u}_j$  and  $\mathbf{u}_{j+1}$  form a pair of green directions and  $\mathbf{u}_{k+1}$  is a single blue direction (five directions in total). Note that in this case we have  $u_j = w_2(z')$  and  $u_{k+1} = w_3(z')$ .
  - (c)  $z_j = z_k = z'$ : Here we have a *unique* direction set where  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  are the two red directions:  $\mathbf{u}_j$  and  $\mathbf{u}_{j+1}$  form a pair of green directions and  $\mathbf{u}_k$  and  $\mathbf{u}_{k+1}$  form a pair of blue directions (six directions in total).

The above algorithm clearly takes  $O(\sigma)$  time, since steps 1 and 2 take constant time, and are iterated at most  $2\sigma$  times: in each iteration  $j$  and/or  $k$  is increased, and there are at most  $\sigma$  possibilities for each iterator.

**Theorem 4.2** *The direction sets (one or two) for a given pair of red directions can be determined in  $O(\sigma)$  time.*

#### 4.2 Linear-time enumeration algorithm

The idea of the linear-time algorithm to enumerate all coloured direction sets is first to compute a set of supporting lines  $l_1, l_2$  and  $l_3$  for a fixed pair of red directions  $\mathbf{u}_{i-1}$  and  $\mathbf{u}_i$  using the  $O(\sigma)$  time algorithm from the previous section. Intuitively, we then *rotate* all three supporting lines in counter-clockwise order around  $\partial\mathcal{D}$  while maintaining the centroid-property, locating all positions of the supporting lines that correspond to direction sets. In order to prove that this can be done efficiently, we need some definitions and technical results.

Consider a line  $l_1$  that supports  $\partial\mathcal{D}$  at a single vertex  $u_i$ . We say that  $u_i$  is the rotation point of  $l_1$ . Consider a counter-clockwise rotation of  $l_1$  around its rotation point. We continue this rotation until  $l_1$  also supports the successor  $u_{i+1}$  of  $u_i$ . Then  $u_{i+1}$  becomes the new rotation point, and we continue the counter-clockwise rotation. This is called a rotation of  $l_1$  around  $\partial\mathcal{D}$ .

Each step in the algorithm will strictly rotate around  $\partial\mathcal{D}$  at least one of the supporting lines  $l_1, l_2$  and  $l_3$  that fulfill the centroid-property. However, not all supporting lines necessarily

rotate in every step. The important fact that we will prove in Lemma 4.2 below is that none of the supporting lines need to rotate *clockwise* in order to maintain the centroid-property. Furthermore, in each step of the algorithm—which takes constant time—at least one of the supporting lines will change its rotation point to the successor on  $\partial\mathcal{D}$  of the previous rotation point; such a supporting line will after the rotation step be a *tangent* supporting both the previous and new rotation point. Thus the running time of the algorithm is clearly  $O(\sigma)$ , since each supporting line has exactly  $2\sigma$  possible rotation points.

Two types of constant-time rotation steps are used by the algorithm where the latter is only performed if the former cannot be performed.

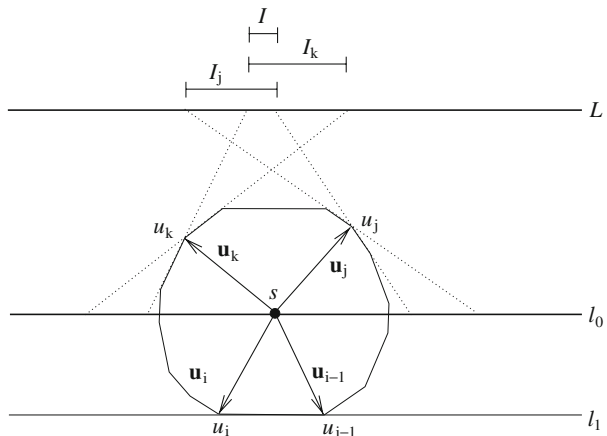
*Degenerate rotation.* Consider any of the current supporting lines  $l_1, l_2$  and  $l_3$ . If this supporting line is part of more than one set of supporting lines fulfilling the centroid-property, then we have the situation considered in case 1 in the proof of Lemma 3.5 and illustrated in Fig. 4b. In this case we simply rotate the other two supporting lines counter-clockwise as much as possible; in Fig. 4b the situation after this rotation corresponds to the case where one of the lines supports  $\partial\mathcal{D}$  at  $u_j$  and  $u_{j+1}$ , while the other line only supports  $\partial\mathcal{D}$  at  $u_k$ .

*Non-degenerate rotation.* Consider the current supporting lines and assume that their rotation points are  $u_i, u_j$  and  $u_k$ , respectively. Imagine that we rotate  $l_1$  maximally counter-clockwise while maintaining the centroid-property and the invariant that  $l_1, l_2$  and  $l_3$  support  $\partial\mathcal{D}$  at  $u_i, u_j$  and  $u_k$ , respectively. In Lemma 4.2 we prove that this *cannot* force  $l_2$  and  $l_3$  to rotate clockwise around their rotation points (i.e., in the “opposite” direction). A rotation of  $l_1$  therefore must lead to one of the following events:

- $l_1$  becomes a tangent through  $u_i$  and  $u_{i+1}$
- $l_2$  becomes a tangent through  $u_j$  and  $u_{j+1}$
- $l_3$  becomes a tangent through  $u_k$  and  $u_{k+1}$

Consider the event that  $l_1$  becomes a tangent through  $u_i$  and  $u_{i+1}$ . We know that  $l_2$  and  $l_3$  must support  $\partial\mathcal{D}$  at  $u_j$  and  $u_k$ , respectively. Thus we need to determine if there exists a set of supporting lines fulfilling the centroid-property, where  $l_1$  is a tangent through  $u_i$  and  $u_{i+1}$ ,  $l_2$  supports  $\partial\mathcal{D}$  at  $u_j$  and  $l_3$  supports  $\partial\mathcal{D}$  at  $u_k$ . This problem can be solved in constant time using a simple geometric construction as shown in Fig. 7. The interval  $I = I_j \cap I_k$  on  $L$  must be non-empty, and among the points  $z \in I$  there should be one for which the supporting lines through  $z$  have identical distances on  $l_0$  to  $s$ . This can be tested in constant time by

**Fig. 7** Constant-time construction to determine if there exists a set of supporting lines fulfilling the centroid-property, where  $l_1$  is a tangent through  $u_i$  and  $u_{i+1}$ ,  $l_2$  supports  $\partial\mathcal{D}$  at  $u_j$  and  $l_3$  supports  $\partial\mathcal{D}$  at  $u_k$



constructing the supporting lines for the two endpoints of  $I$ . (Note that we need not compute the correct location of  $z$ , but only show that it exists.)

By using a similar constant-time construction we can test if there exists a set of supporting lines fulfilling the centroid-property, where  $l_1$  supports  $\partial\mathcal{D}$  at  $u_i$ ,  $l_2$  is a tangent through  $u_j$  and  $u_{j+1}$ , and  $l_3$  supports  $\partial\mathcal{D}$  at  $u_k$ . Finally, we can also test in constant-time if there exists a set of supporting lines fulfilling the centroid-property, where  $l_1$  supports  $\partial\mathcal{D}$  at  $u_i$ ,  $l_2$  supports  $\partial\mathcal{D}$  at  $u_j$ , and  $l_3$  is a tangent through  $u_k$  and  $u_{k+1}$ . If more than one set of supporting lines exist, we choose one among the sets in lexicographical order, that is, choose the set where  $l_1$  is a tangent first, then the one where  $l_2$  is a tangent and finally the one where  $l_3$  is a tangent.

**Lemma 4.2** *If we rotate  $l_1$  counter-clockwise while maintaining the centroid-property and the invariant that  $l_1, l_2$  and  $l_3$  support  $\partial\mathcal{D}$  at  $u_i, u_j$  and  $u_k$ , respectively, then neither  $l_2$  nor  $l_3$  can rotate in clockwise direction.*

*Proof* Assume to the contrary that as we rotate  $l_1$  counter-clockwise,  $l_2$  rotates (strictly) clockwise. Let  $(l_1^A, l_2^A, l_3^A)$  be the set of supporting lines before we started this rotation. As we continue the rotation of  $l_1$  (possibly via succeeding rotation points) around  $\partial\mathcal{D}$ , we reach a point where  $l_2$  is “pushed back” to its original position. That is, we have another a set of supporting lines  $(l_1^B, l_2^B, l_3^B)$  that fulfils the centroid-property where  $l_2^B = l_2^A$ . But since  $l_1^B$  is rotated strictly more counter-clockwise than  $l_1^A$ , this a contradiction to the fact that we should have rotated  $l_1^A$  in a *degenerate rotation step* when we had the set of supporting lines  $(l_1^A, l_2^A, l_3^A)$ .

A similar argument can be used to prove that  $l_3$  cannot rotate clockwise as we rotate  $l_1$  counter-clockwise.  $\square$

**Theorem 4.3** *The set of coloured direction sets can be determined in  $\Theta(\sigma)$  time.*

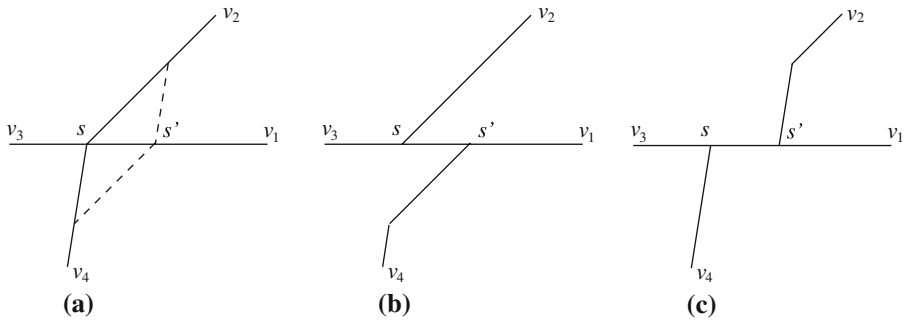
## 5 Directions in a full SMT

The aim of this section is to show that the edges in a full SMT use at most six legal orientations. This will prove to be a powerful result for developing canonical forms in later parts of the paper. Steiner points in any normed plane have degree  $m = 3$  or  $m = 4$  [11, 29]. In this section we first show that if the full SMT is fulsome, then we may assume that all Steiner points have degree  $m = 3$  (except in a very special case). Then we prove that there exists a single direction set that is used by every (degree three) Steiner point in a full and fulsome SMT.

### 5.1 Splitting of degree four Steiner points

A degree four Steiner point  $s$  consists of two opposite pairs of edges. Let the neighbours of  $s$  be denoted by  $v_1, v_2, v_3$  and  $v_4$  in counter-clockwise order around  $s$  (Fig. 8a). One of the opposite pairs of edges, say,  $(s, v_1)$  and  $(s, v_3)$  must be *collinear* [29]; this is called the *first* pair of opposite edges. The other pair of edges,  $(s, v_2)$  and  $(s, v_4)$ , is called the *second* pair of opposite edges. Our classification into pairs is not necessarily unique, but this is not important in the following. The results of the following two lemmas were essentially given by Li et al. [20] (their Lemma 2); somewhat shorter proofs based on the work by Swanepoel [29] are given in [7].

**Lemma 5.1** *The first pair of opposite edges around a degree four Steiner point must be straight.*



**Fig. 8** Degree four Steiner point. (a) Movement of  $s$  to point  $s'$  on the first pair of opposite edges. (b) Splitting into degree three Steiner points (option one). (c) Splitting into degree three Steiner points (option two)

The second pair of opposite edges has a slightly relaxed, but still quite restricted form:

**Lemma 5.2** *The second pair of opposite edges around a degree four Steiner point uses adjacent legal orientations only. Furthermore, let  $S$  be the intersection of the shortest path parallelogram  $R(v_2, v_4)$  with the first pair of opposite edges (hence  $S$  is a segment). Then  $|v_2s'|_{\mathcal{D}}$  is the same for any point  $s' \in S$ ; similarly,  $|v_4s'|_{\mathcal{D}}$  is the same for any point  $s' \in S$ .*

A consequence of this lemma is that if the second pair of opposite edges are not straight and collinear, then we can always split the Steiner point  $s$  into a pair of adjacent degree three Steiner points. In fact, this split can be made in two topologically different ways as indicated in Fig. 8b and c.

Define a *cross* as a degree four Steiner point where both the first and second pair of opposite edges are straight and collinear. So far we have shown that unless the degree four Steiner point is a cross, we can always split it into two adjacent degree three Steiner points. We will now show that even if the Steiner point is a cross, then in most cases we can still split it into two degree Steiner points. First we prove an intermediate result that turns out to be helpful in later parts of the paper.

**Lemma 5.3** *Let  $e = (s_1, s_2)$  be an edge connecting two Steiner points ( $s_1$  and  $s_2$ ) in a fullsome SMT  $T$ . Let  $e_1 = (s_1, v_1)$  be the next edge incident with  $s_1$  travelling counter-clockwise from  $e$ , and let  $e_2 = (s_2, v_2)$  be the next edge incident with  $s_2$  travelling clockwise from  $e$ . Then there exists an embedding of  $T$  such that  $\theta$ , the angle at  $s_1$  between  $e$  and  $e_1$ , and  $\phi$ , the angle at  $s_2$  between  $e$  and  $e_2$ , satisfy  $\theta + \phi > \pi$ .*

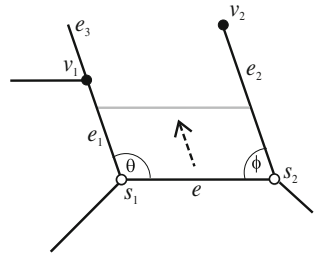
*Proof* For the given embedding of  $T$ , let  $\mathbf{u}_1$  be the outward direction of  $e_1$  at  $s_1$  and let  $\mathbf{u}_2$  be the outward direction of  $e_2$  at  $s_2$ . Observe that

$$\theta + \phi \geq \pi \tag{1}$$

since otherwise we could simultaneously perturb  $s_1$  in direction  $\mathbf{u}_1$  and  $s_2$  in direction  $\mathbf{u}_2$  decreasing  $|s_1s_2|$  and hence decreasing  $|T|_{\mathcal{D}}$ , which contradicts minimality.

Now suppose that the lemma does not hold; in other words, that  $\theta + \phi = \pi$ , as illustrated in Fig. 9. Then we can perform a *slide* on  $e$ , by which we mean a simultaneous movement of  $s_1$  in direction  $\mathbf{u}_1$  and  $s_2$  in direction  $\mathbf{u}_2$  (at the same speed), without increasing the length of  $T$ . (This is an example of a *zero-shift*, which we study in more detail in Sect. 6.) Note that neither  $e_1$  nor  $e_2$  contains a corner point, since otherwise there would exist an alternative

**Fig. 9** A slide on the edge  $e$



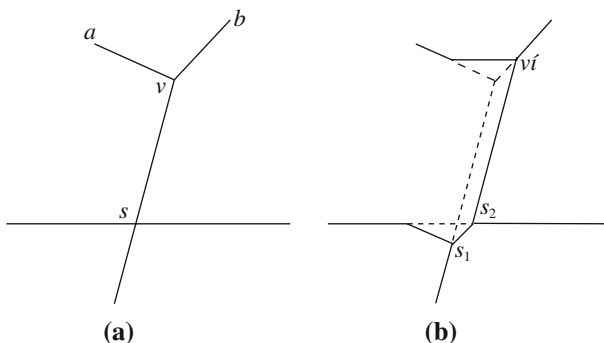
embedding of the edge, which, by Eq. 1, would have a larger angle at  $s_1$  or  $s_2$ . It follows that we can continue to slide  $e$  until either  $s_1$  coincides with  $v_1$  or  $s_2$  coincides with  $v_2$ .

Assume, without loss of generality, that  $s_1$  coinciding with  $v_1$  occurs first. If  $v_1$  is a terminal then we have a contradiction to  $T$  being fulsome. If  $v_1$  is a Steiner point then let  $\theta_1$  be the meeting angle at  $v_1$  between  $e_1$  and the next edge  $e_3$  going counter-clockwise around  $v_1$  from  $e_1$ . By Theorem 3.1,  $\theta_1 \leq \pi$ . If  $\theta_1 < \pi$  then there exists an embedding of  $e_3$  such that continuing to slide  $e$  past  $v_1$  strictly decreases the length of  $e$ , contradicting the minimality of  $T$ . Hence  $\theta_1 = \pi$ , and  $e_3$  is a straight edge. We can now continue to slide  $e$  past  $v_1$  without increasing  $|T|_{\mathcal{D}}$ . The same argument applies at each Steiner point encountered. Hence we can continue the slide until we reach a terminal, giving a contradiction to  $T$  being fulsome.  $\square$

A consequence of this lemma is that in a fulsome SMT we cannot have a pair of parallel straight edges that are connected by a third edge.

**Theorem 5.1** *In a fulsome SMT, a degree four Steiner point can always be split into two adjacent degree three Steiner points unless it is a cross and is adjacent to terminals only.*

*Proof* By the comment following Lemma 5.2, we only need to consider the case where the Steiner point  $s$  is a cross. Assume that one of the neighbours of  $s$  is a Steiner point  $v$ , and that  $v$  has degree three. By Lemma 5.3, there exists an embedding of the SMT such that neither of the line segments  $va$  and  $vb$  connecting  $v$  to the two neighbours other than  $s$  are parallel to the straight edges incident to  $s$  (Fig. 10a). Hence we can make the (small) local change indicated in Fig. 10b without increasing the length of the tree. This change will split  $s$  into two adjacent degree three Steiner points  $s_1$  and  $s_2$  while moving  $v$  to a new position  $v'$ .



**Fig. 10** Splitting of a cross into two adjacent degree three Steiner points



Now, if  $v$  was a degree four Steiner point, then we arrive at a contradiction to length-minimality, since the local change given above would decrease the length of the tree.  $\square$

In the remainder of this paper we will therefore assume that there always exists an embedding of a full and fulsome SMT where all Steiner points have degree three. (The construction of a cross with terminals as neighbours can easily be handled separately.)

### 5.2 Direction sets for full SMTs

Let  $T$  be a full and fulsome SMT where all Steiner points have degree three. We begin by describing a (not necessarily unique) method of colouring the edges of  $T$ .

Pick any Steiner point  $s$  and any feasible coloured direction set  $\mathcal{U}$  for  $s$ . The direction set  $\mathcal{U}$  defines a colour for each of the edges incident with  $s$ , and these appear as red, green and blue in counter-clockwise order around  $s$ . Now pick any Steiner point neighbour  $s'$  of  $s$ . Again assume that the colours appear in counter-clockwise order as red, green and blue around  $s'$ ; thus the single coloured edge incident with  $s'$  uniquely defines the colours of the other two edges. Repeat this procedure until all edges of  $T$  have been coloured.

We also assign a parity to the vertices of  $T$  depending on whether the path in  $T$  from  $s$  to that vertex contains an odd or even number of edges. Since  $T$  is a tree, this assignment of parity is well defined. When we in the following say that there exists a single direction set  $\mathcal{U}$  that is used by all Steiner points of  $T$ , the interpretation should be that the direction set  $\mathcal{U}$  is used at even vertices while the complementary direction set of  $\mathcal{U}$  is used at odd vertices.

**Theorem 5.2** *Given a full and fulsome SMT, there exists a single direction set that is used by every Steiner point in the tree.*

*Proof* We prove the theorem by showing that, given any two adjacent Steiner points  $s_1$  and  $s_2$  in a full and fulsome SMT, there exists a single direction set that is used by  $s_1$  and  $s_2$ , and furthermore there exists a small finite perturbation of  $s_1$  and  $s_2$  such that the resulting tree is still a full SMT and the directions used by the edges incident with  $s_1$  exactly coincide with the directions used by the edges incident with  $s_2$ . This means that the direction set at any Steiner point can be propagated throughout the SMT, since all internal nodes of the full SMT are Steiner points. The theorem then immediately follows.

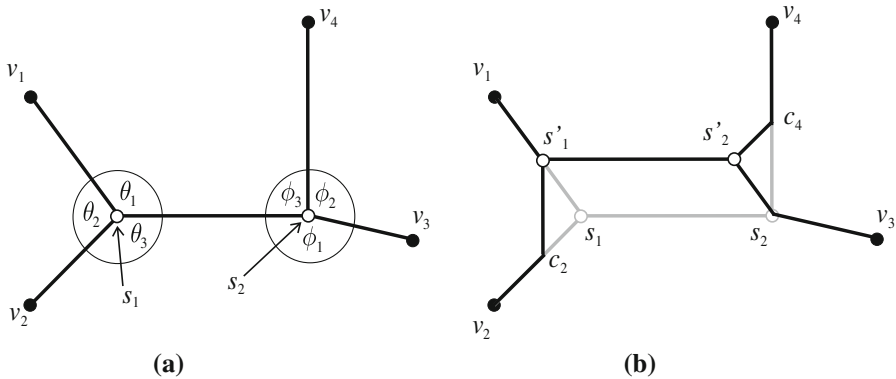
Let  $s_1$  and  $s_2$  be adjacent Steiner points in a full and fulsome SMT. For a given embedding of this SMT, let  $v_1$  and  $v_2$  be the nodes or corner points adjacent to  $s_1$  on the other two incident edges, travelling counter-clockwise from  $s_1s_2$ , i.e., line segments  $s_1v_1$  and  $s_1v_2$  each use a single legal orientation. Similarly, let  $v_3$  and  $v_4$  be the nodes or corner points adjacent to  $s_2$  on the other two incident edges, travelling counter-clockwise from  $s_2s_1$ . Let  $T$  be the resulting full SMT on terminal set  $\{v_1, v_2, v_3, v_4\}$  with Steiner points  $s_1$  and  $s_2$ . This is illustrated in Fig. 11a. Let  $\mathbf{u}_1$  and  $\mathbf{u}_2$  be the directions of  $\overrightarrow{s_1v_1}$  and  $\overrightarrow{s_1v_2}$ , respectively, and let  $\mathbf{u}_3$  and  $\mathbf{u}_4$  be the directions of  $\overrightarrow{s_2v_3}$  and  $\overrightarrow{s_2v_4}$ , respectively.

For simplicity, we assume either that  $s_1s_2$  is a straight edge or that it is embedded using two corner points (as in Fig. 2b), so that both ends of the edge use the same direction. Let  $\theta_1, \theta_2, \theta_3$  be the three angles around  $s_1$ , travelling counter-clockwise from  $s_1s_2$ , and let  $\phi_1, \phi_2, \phi_3$  be the three angles around  $s_2$ , travelling counter-clockwise from  $s_2s_1$ . Again, this is illustrated in Fig. 11a. By Lemma 5.3, we can choose the embedding of the original full SMT such that

$$\theta_1 + \phi_3 > \pi. \tag{2}$$

Suppose that

$$\theta_3 > \phi_3. \tag{3}$$



**Fig. 11** Performing a shift on adjacent Steiner points

Under this assumption, we construct a transformation on  $T$  that does not increase its length. We do this by defining a *shift* on  $s_1$  and  $s_2$  (shifts will be discussed in more generality in the next section). This involves moving each of  $s_1$  and  $s_2$  a distance  $\varepsilon$  in direction  $\mathbf{u}_1$  to  $s'_1$  and  $s'_2$ , respectively, where  $\varepsilon > 0$  is small compared to all edge lengths in  $T$ .

We claim that we can construct a line segment from  $s'_1$  in direction  $-\mathbf{u}_4$  meeting  $s_1 v_2$  at a point  $c_2$ . For such a construction to be possible, for sufficiently small  $\varepsilon$ , we require that the ray from  $s_1$  in direction  $\mathbf{u}_2$  must intersect the ray from  $s'_1$  in direction  $-\mathbf{u}_4$ . This occurs if  $\phi_3 \geq \pi - \theta_1$  and  $\phi_3 < \theta_3$  which follow from inequalities (2) and (3), respectively. Similarly, we can construct a line segment from  $s'_2$  in direction  $-\mathbf{u}_2$  meeting  $s_2 v_4$  at a point  $c_4$ . Furthermore,  $c_4$  does not coincide with  $s'_2$  since inequality (2) is a strict inequality.

Now, construct a new tree  $T'$  interconnecting  $\{v_1, v_2, v_3, v_4\}$  via Steiner points  $s'_1$  and  $s'_2$ , such that  $c_2$  is the corner point of the edge  $s'_1 v_2$ ,  $c_4$  is the corner point of the edge  $s'_2 v_4$  and  $s_2$  is the corner point of the edge  $s'_2 v_3$ . The remaining external edge,  $s'_1 v_1$ , is a straight edge. This is illustrated in Fig. 11b.

We next observe that  $|T'|_{\mathcal{D}} = |T|_{\mathcal{D}}$ . To see this, note that in transforming  $T$  to  $T'$  the edge  $s_1 s_2$  and line segments  $s_1 s'_1$ ,  $s_1 c_2$  and  $s_2 c_4$  have been removed, and the edge  $s'_1 s'_2$  and line segments  $s_2 s'_2$ ,  $s'_2 c_4$  and  $s'_1 c_2$  have been added. But  $|s_1 s_2|_{\mathcal{D}} = |s'_1 s'_2|_{\mathcal{D}}$  (since  $s_1$  and  $s_2$  undergo identical translations),  $|s_1 s'_1|_{\mathcal{D}} = |s_2 s'_2|_{\mathcal{D}}$  (by construction) and  $|s_1 c_2|_{\mathcal{D}} = |s'_2 c_4|_{\mathcal{D}}$ ,  $|s_2 c_4|_{\mathcal{D}} = |s'_1 c_2|_{\mathcal{D}}$  (since triangles  $\Delta s_1 s'_1 c_2$  and  $\Delta s'_2 s_2 c_4$  are congruent). Hence,  $|T'|_{\mathcal{D}} = |T|_{\mathcal{D}}$ . In other words,  $T'$  is also a full SMT on the terminal set of  $T$ .

We now show that there is a single direction set that is used by  $s_1$  and  $s_2$  in the original full SMT, and that by performing a pair of shifts as described above we can construct a new SMT such that the directions used by the edges incident with  $s_1$  exactly coincide with the directions used by the edges incident with  $s_2$ . We do this by showing that, for each of the three colour labels for edges, the direction sets for  $s_1$  and  $s_2$  coincide on that colour and we can ensure that all directions are used at each Steiner point. Note that this is trivially true for the colour of the edge  $s_1 s_2$ . Now, observe that the edges (or half-edges)  $s_1 v_2$  and  $s_2 v_4$  are labelled with the same colour (say, blue). If both these line segments use the same direction for every embedding then the direction sets for  $s_1$  and  $s_2$  coincide on the blue colour. If there exists an embedding such that  $s_1 v_2$  and  $s_2 v_4$  have different directions then either  $\theta_3 > \phi_3$  or  $\phi_3 > \theta_3$ . In either case we can perform the local transformation above (swapping the roles of  $s_1$  and  $s_2$  if necessary) resulting in a new SMT that uses both blue directions at  $s'_1$  and  $s'_2$ . In order for this new tree to be minimal (under any embedding of the original tree) it again

follows that the direction sets for  $s_1$  and  $s_2$  coincide on the blue label, and that both directions are used by both edges after applying the transformation. The same argument applies to the remaining colour label, concluding the proof.  $\square$

The statement of this theorem is similar in spirit to the result of Du et al. [12] for metrics defined by strictly convex and differentiable unit circles, which says that edges of a full SMT use exactly three different orientations. A corollary of our theorem is that the edges of a full and fulsome SMT for a metric defined by a polygonal unit circle use at most six legal orientations. In the following section we show that four legal orientations actually suffice.

### 6 Length-preserving shifts and canonical forms

The efficiency of the algorithms that we develop in this paper for constructing SMTs comes from the fact that we can assume that full SMTs have particular canonical forms. Our means of establishing these canonical forms is to use the properties of length preserving perturbations, denoted zero-shifts, which we describe in this section (see also [13, 17–19]).

Given a weighted set of legal orientations defining a normed plane, let the ordered subset  $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k\}$  be a *coloured* direction set, as defined in Sect. 4. The elements of  $\mathcal{U}$ , treated as vectors rooted at the origin, appear in counterclockwise order, beginning with  $\mathbf{u}_1$ , which corresponds to the exclusively primary red direction, and  $\mathbf{u}_2$ , which corresponds to the exclusively secondary red direction. By Lemma 4.1 we know that  $k = 4, 5$  or  $6$ . We define the *direction weight set* of  $\mathcal{U}$  to be the set  $\{w_1, w_2, \dots, w_k\}$  where each  $w_i = 1/|\mathbf{u}_i|$  (where the norm  $|\cdot|$  is the usual Euclidean norm). We define the *direction angle set* of  $\mathcal{U}$  to be the set  $\{\theta_1, \theta_2, \dots, \theta_k\}$  where each  $\theta_i$  is the angle between  $\mathbf{u}_i$  and  $\mathbf{u}_{i+1}$  (where addition in the subscripts is modulo  $k$ ). Hence, for each  $i \in \{1, \dots, k\}$  we have  $\cos \theta_i = w_i w_{i+1} (\mathbf{u}_i \cdot \mathbf{u}_{i+1})$ . Note that the direction angle set must also satisfy the condition that  $\sum_{i=1}^k \theta_i = 2\pi$ .

#### 6.1 Fundamental zero-shifts and their applications

For a given coloured direction set  $\mathcal{U}$ , let  $T$  be a full and fulsome SMT. Furthermore, assume that the edges of  $T$  uses all directions in  $\mathcal{U}$ .

We define a *zero-shift* as a perturbation of one or more Steiner points in  $T$  such that the perturbation does not increase the length of  $T$ . Such a perturbation  $\mathbf{v}$  is called a *fundamental zero-shift* if it cannot be decomposed into two zero-shifts each of which acts on a subset of the Steiner points acted on by  $\mathbf{v}$ , and at least one of which acts on a proper subset of those Steiner points. Here we will investigate those fundamental zero-shifts that act on either one Steiner point or two adjacent Steiner points in  $T$ . In Sect. 6.2 we will in fact show that these are the only fundamental zero-shifts that can occur in  $T$ .

Let  $s$  be a Steiner point in  $T$  such that the edges incident with  $s$  use all directions in  $\mathcal{U}$ . Clearly, for any  $\mathcal{U}$ , there exists a  $T$  that contains such a Steiner point; for example, we can construct a suitable SMT  $T$  with a single Steiner point.

We now consider properties of 1-point fundamental zero-shifts for  $s$ . We will show that such a perturbation can exist only under very special conditions.

**Lemma 6.1** *Suppose the direction set  $\mathcal{U}$  of  $T$  contains exactly 4 directions. Then there are no 1-point zero-shifts for any Steiner points in  $T$ .*

*Proof* Let  $s$  be a Steiner point in  $T$ . We first show that the angle  $\theta$  between the blue and green directions is strictly less than  $\pi$ .

Suppose, on the contrary, that  $\theta = \pi$ . Consider the SMT  $T'$  consisting of the edges in  $T$  incident with  $s$  and their end points, where the terminals,  $v_1, v_2, v_3$ , of  $T'$  are the three neighbours of  $s$ . Let  $(v_1, s)$  be the red edge of  $T'$ . Since  $\theta = \pi$ , it follows that  $v_2, s$  and  $v_3$  are collinear. By a continuity argument, similar to the argument in the proof of Lemma 4.1, a sufficiently small perturbation of  $v_2$  off the line through  $v_2$  and  $v_3$  will result in only a small bounded perturbation of  $s$ . After applying this perturbation, the green and blue edges will use their existing directions but require at least one extra direction in order to intersect close to  $s$ . This means that  $\mathcal{U}$  contains at least five directions, giving a contradiction.

Hence,  $0 < \theta < \pi$ . It follows that the position of  $s$  is uniquely determined by the positions of the two adjacent vertices incident with the green and blue edges, and thus there can be no 1-point zero-shift of  $s$ .  $\square$

If  $\mathcal{U}$  contains five directions then the following lemma holds [7]:

**Lemma 6.2** *Suppose  $\mathcal{U}$ , the direction set for  $T$ , contains exactly five directions, i.e.,  $\mathcal{U} = \{\mathbf{u}_1, \dots, \mathbf{u}_5\}$ . Let  $s$  be a Steiner point in  $T$  using all directions in  $\mathcal{U}$ . Without loss of generality, we assume that there are two red directions  $(\mathbf{u}_1, \mathbf{u}_2)$  and two green directions  $(\mathbf{u}_3, \mathbf{u}_4)$ . Let  $\{w_1, w_2, \dots, w_5\}$  and  $\{\theta_1, \theta_2, \dots, \theta_5\}$  be the direction weight set and direction angle set, respectively, of  $\mathcal{U}$ . Then there exist 1-point fundamental zero-shifts for  $s$ , perturbing  $s$  in the directions of  $\mathbf{u}_5$  and  $-\mathbf{u}_5$ .*

Note that Lemma 6.2 still applies if any of the meeting angles are  $\pi$ . Indeed, in some cases we get slightly stronger results. If  $\theta_3 + \theta_4 = \pi$ , then it is clear from the degeneracy of the zero-shift that the perturbation can be performed in either direction even if no edge incident with  $s$  uses direction  $\mathbf{u}_4$ . Similarly, if  $\theta_1 + \theta_5 = \pi$ , then a 1-point fundamental zero-shift can be performed in either direction even if no edge incident with  $s$  uses direction  $\mathbf{u}_1$ .

If  $\mathcal{U}$  contains six directions then we get the following result [7]:

**Lemma 6.3** *Suppose  $\mathcal{U} = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_6\}$  is the direction set for  $T$ . Let  $s$  be a Steiner point in  $T$  using all directions in  $\mathcal{U}$ . Let  $\{w_1, w_2, \dots, w_6\}$  and  $\{\theta_1, \theta_2, \dots, \theta_6\}$  be the direction weight set and direction angle set, respectively, of  $\mathcal{U}$ . Then for every direction, there exists a 1-point fundamental zero-shift for  $s$ , perturbing  $s$  in that direction.*

In addition to the properties given by Lemmas 6.2 and 6.3, precise conditions exist for determining the existence of direction sets with cardinality 5 or 6 in terms of the directions and weights [7].

Next, we consider properties of 2-point fundamental zero-shifts for  $T$ . The results here are very similar to those in [6], and hence only require a brief discussion.

**Lemma 6.4** *Let  $s_1$  and  $s_2$  be neighbouring Steiner points in  $T$ . Assume that  $(s_1, s_2)$  is a straight edge which is neither exclusively primary nor exclusively secondary. Let  $e_1$  and  $e_2$  be distinct edges of  $T$  incident with  $s_1$  and  $s_2$  respectively, such that  $e_1$  and  $e_2$  have the same colour. Assume that for at least one of  $i = 1$  or  $i = 2$  the meeting angle between the two edges other than  $e_i$  at  $s_i$  is not  $\pi$ . For each  $i \in \{1, 2\}$  let  $v_i$  be the closest neighbouring node or corner point on  $e_i$  to  $s_i$ . If  $\overrightarrow{s_1 v_1}$  and  $\overrightarrow{s_2 v_2}$  have different directions then there exists a 2-point fundamental zero-shift for  $s_1$  and  $s_2$ .*

This lemma follows immediately from the proof of Theorem 5.2, using the construction illustrated in Fig. 11. Note that the condition on  $(s_1, s_2)$  means that the direction set of  $T$  contains at most five directions. It follows that there are no 1-point zero-shifts that move  $s_1$

or  $s_2$  in the same direction as the constructed 2-point zero-shift, and hence that the 2-point zero-shift is fundamental.

The remaining 2-point zero-shift not covered by Lemma 6.4 is where  $T$  necessarily contains meeting angles of  $\pi$  at both  $s_1$  and  $s_2$ , in each case between the two incident edges other than  $e_1$  or  $e_2$ . This implies that all edges incident with  $s_1$  and  $s_2$  other than  $e_1$  and  $e_2$  are straight and collinear. In this case, the resulting 2-point zero-shift is no longer fundamental, but can be decomposed into two 1-point zero-shifts, by the same argument used in the proof of Lemma 5.2.

We conclude this section by proving two important properties of full and fulsome SMTs that follow from the above properties of fundamental zero-shifts.

**Lemma 6.5** *Let  $s$  be a Steiner point in a fulsome SMT  $T$ . Let  $u$ ,  $v$  and  $w$  be the three neighbouring nodes of  $s$  in  $T$ , and suppose the edges  $(s, u)$  and  $(s, v)$  are straight and  $s$ ,  $u$  and  $v$  are collinear. Then  $w$  is a terminal.*

*Proof* Let  $\mathbf{u}$  and  $-\mathbf{u}$  be the directions of the edges  $(s, u)$  and  $(s, v)$ . We prove the lemma by contradiction.

Assume  $w$  is not a terminal, and let  $a$  and  $b$  be the other two neighbouring nodes of  $w$ . Suppose one of these nodes (say,  $a$ ) has the property that the edge between  $w$  and that node,  $(w, a)$ , is straight and has direction  $\mathbf{u}$  or  $-\mathbf{u}$ . Then, by Lemma 5.3, we obtain a contradiction to  $T$  being a fulsome SMT.

If, on the other hand, neither  $(w, a)$  nor  $(w, b)$  is straight with direction  $\mathbf{u}$  or  $-\mathbf{u}$ , then it follows that the direction set for  $T$  contains at least five directions, with two directions for each of the two colours associated with  $(s, u)$  and  $(s, v)$ . Hence, by Lemmas 6.2 and 6.3, there exists a 1-point zero-shift on  $w$  which we can continue to apply until either  $(w, a)$  or  $(w, b)$  is straight with direction  $\mathbf{u}$  or  $-\mathbf{u}$ . Hence we again obtain the same contradiction as above.  $\square$

A consequence of Lemma 6.5 is the following theorem which gives strong restrictions on when degree four Steiner points can occur as part of a fulsome SMT.

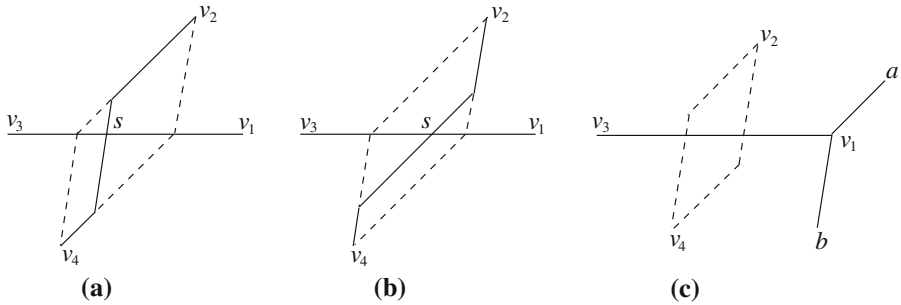
**Theorem 6.1** *In a fulsome SMT, a degree four Steiner point must be a cross, unless it is adjacent to terminals only.*

*Proof* Consider a degree four Steiner point  $s$  with neighbours  $v_1$ ,  $v_2$ ,  $v_3$  and  $v_4$  which does not form a cross, that is, the second pair of opposite edges,  $(s, v_2)$  and  $(s, v_4)$ , are not straight and collinear (as in Fig. 8).

The fact that vertices  $v_2$  and  $v_4$  must be terminals follows by splitting the Steiner point  $s$  into two adjacent degree three Steiner points and applying Lemma 6.5.

Before we prove that  $v_1$  and  $v_3$  also must be terminals, we make a few observations. Let  $S$  be the line segment between  $v_1$  and  $v_3$ . By moving  $s$  along  $S$ , we can always make the second pair of opposite edges *locally collinear* in two different ways as shown in Fig. 12a and b. As a consequence, the vertex  $v_1$  cannot be degree four Steiner point, since then we could construct a pair of locally collinear edges around  $s$  and  $v_1$ , respectively, but the collinear edges of  $s$  and  $v_1$  would be non-parallel—hence making a length-decreasing shift of the edge  $(s, v_1)$  feasible.

So we may assume that  $v_1$  is a degree three Steiner point. By splitting  $s$  into degree three Steiner points in the two topologically different ways shown in Fig. 8b and c, and applying Theorem 5.2, it follows that the two edges  $(v_1, a)$  and  $(v_1, b)$  incident to  $v_1$  (other than  $(v_1, s)$ ) must use the same pair of adjacent legal orientations as  $(s, v_2)$  and  $(s, v_4)$ . Therefore, if one



**Fig. 12** Illustration of observations for proving that a degree four Steiner point in a fulsome SMT must be a cross

of the edges  $(v_1, a)$  or  $(v_1, b)$  is a bent edge, then we can always construct a pair of locally collinear edges at  $v_1$ . This again leads to a contradiction to length-minimality by using the same arguments as in the degree four case.

As a consequence, we are left with the case where the edges  $(v_1, a)$  and  $(v_1, b)$  are straight and not collinear (Fig. 12c). However, it is always possible to move  $s$  along  $S$  (the segment between  $v_1$  and  $v_3$ ) to a point  $s'$  where either  $(s', v_2)$  is straight and parallel to  $(v_1, a)$ —or  $(s', v_4)$  is straight and parallel to  $(v_1, b)$ . In both cases, by Lemma 5.3 we arrive at a contradiction to fulsomeness.  $\square$

### 6.2 General zero-shifts

The following lemma gives conditions for the existence of general zero-shifts, and establishes a key property of such shifts: informally, that they can move exclusively primary or exclusively secondary material from one edge to another. This lemma is the key tool for constructing the canonical forms in Sect. 6.3.

**Theorem 6.2** *Let  $e_1$  and  $e_2$  be two edges in a full and fulsome SMT  $T$  such that  $e_1$  has an exclusively secondary component and  $e_2$  has an exclusively primary component. Then there exists a zero-shift acting on the Steiner points on the path from  $e_1$  to  $e_2$  in  $T$ , such that the shift can continue to be performed until either  $e_1$  has no exclusively secondary component or  $e_2$  has no exclusively primary component. Furthermore, this shift preserves the direction of all straight edges except (possibly)  $e_1$  and  $e_2$ .*

*Proof* Let  $s_1$  and  $s_2$  be, respectively, the first and last Steiner points on the path from  $e_1$  to  $e_2$  in  $T$ . For each  $i \in \{1, 2\}$  let  $\theta_i$  be the meeting angle at  $s_i$  between the two incident edges other than  $e_i$ .

**Case 1** For  $i = 1$  or  $i = 2$  assume that the two edges incident with  $s_i$  are straight edges and  $\theta_i = \pi$ . In that case there is a 1-point zero-shift moving  $s_i$  along the line through those other two edges (as in the proof of Lemma 5.2). Assume first that  $T$  spans at least five terminals. If the moving Steiner point  $s_i$  meets another Steiner point without the edge  $e_i$  becoming straight, then this is a contradiction to fulsomeness by Theorem 6.1. Next assume that  $T$  spans four terminals or less. If a Steiner point  $s_i$  meets another Steiner point, then in fact we must have  $s_1 = s_2$  (the tree  $T$  is a star). In this case we can move the overlapping Steiner points  $s_1$  and  $s_2$  jointly as indicated in the proof of Theorem 6.1 (see also Fig. 12). Hence it is always possible either to make  $e_1$  exclusively primary or to make  $e_2$  exclusively secondary.

**Case 2** If Case 1 does not apply then we argue by induction (as in Theorem 4.1 of [6]). As a first step, we show that the theorem holds for those cases where there are at most two Steiner points in the path from  $e_1$  to  $e_2$ , in other words, for 1-point and 2-point zero-shifts. We also show that in each of these cases the zero-shift strictly decreases the ratio of secondary material to primary material in  $e_1$  and strictly increases it in  $e_2$ .

*Case 2.1: 1-point zero-shifts.* This case occurs where  $e_1$  and  $e_2$  are both incident with the same Steiner point  $s$ , that is,  $s_2 = s_1 = s$ . Let  $v$  be the neighbouring vertex to  $s_1$  that is not an endpoint of  $e_1$  or  $e_2$ . The zero-shift either moves  $s$  away from or towards  $v$ . If  $s$  moves away from  $v$  then the lemma clearly holds. If  $s$  moves towards  $v$  then the only problem that can occur is that the moving Steiner point may meet  $v$  before either  $e_1$  or  $e_2$  is straight. But in that case  $T$  is either not fulsome (if  $v$  is a terminal) or we have two adjacent bent edges incident with a degree four Steiner point, which is a contradiction to  $T$  being an SMT, by Lemma 5.1.

*Case 2.2: 2-point zero-shifts.* This case occurs where  $e_1$  and  $e_2$  are incident with neighbouring Steiner points,  $s_1$  and  $s_2$ . Firstly, suppose that  $e_1$  and  $e_2$  both have the same colour. Then the 2-point zero-shift is a fundamental zero-shift, of the sort illustrated in Fig. 11b (where  $(s_1, v_2) = e_1$  and  $(s_2, v_4) = e_2$ ). This zero-shift acts to decrease the length of one of the edges  $e_0$  incident with  $s_1$  and  $s_2$  but not on the path from  $e_1$  to  $e_2$  (where  $(s_1, v_1) = e_0$  in the figure). It immediately follows that the lemma holds, unless the length of  $e_0$  decreases to 0 before either  $e_1$  or  $e_2$  is straight. However, in such a case the tree contains a degree four Steiner point with an incident bent edge, and a second Steiner point, giving a contradiction to fulsomeness by Theorem 6.1.

Suppose, on the other hand, that  $e_1$  and  $e_2$  have different colours. Let  $e_0$  be the edge incident with  $s_1$  with the same colour as  $e_2$ . If  $e_0$  is exclusively primary, then an appropriate zero-shift can be constructed as follows. We can apply a small 1-point zero-shift at  $s_1$ , effectively transferring an arbitrarily small exclusively secondary component to  $e_0$ . More specifically, it strictly reduces the secondary/primary material ratio in  $e_1$ , and increases it in  $e_0$ . Now there exists a fundamental 2-point zero-shift between the secondary component of  $e_0$  and the edge  $e_2$ . Because the secondary component of  $e$  can be assumed to be arbitrarily small, it follows that this second shift can reduce the secondary component of  $e_0$  to 0. Hence, together these two shifts give a zero-shift on the path between  $e_1$  and  $e_2$ , that preserves the direction of  $e_0$ . The only remaining subcase is where  $e_0$  has an exclusively secondary component, in which case we apply a similar argument, but reverse the order of the two fundamental zero-shifts.

We now conclude the argument by induction on the number of Steiner points in the path between  $e_1$  and  $e_2$ , using the method of proof of Theorem 4.1 of [6], which essentially generalises the construction in the previous paragraph. An important thing to note, in the inductive step, is that we can always find an edge  $e_0$  with exclusively primary or exclusively secondary material and incident with some  $s_0$  on the path between  $s_1$  and  $s_2$  such that the condition of Case 1 ( $\theta_0 = \pi$ ) does not apply at  $s_0$ . This means that the required increase and decrease in secondary/primary material ratio occurs in the smaller zero-shift by the inductive assumptions. The induction argument now follows easily.  $\square$

Note that an immediate corollary of the proof of Theorem 6.2 is that any zero-shift, other than the fundamental zero-shifts described in Sect. 6.1, can be decomposed into two zero-shifts at least one of which acts on a strictly smaller set of Steiner points. Hence, the only fundamental zero-shifts are those described in Sect. 6.1.

### 6.3 Canonical forms

The existence of zero-shifts makes it possible to transform any full and fulsome SMT into another SMT with the same topology that has some well-defined canonical form. In this section we outline the canonical forms proposed in [5] for the uniform orientation metric and show that they also apply to Steiner trees for the fixed orientation metric.

**Definition 6.1** Given an ordering of the edges in a full Steiner topology  $\mathcal{T}$ , an SMT  $T$  for topology  $\mathcal{T}$  is said to be *canonical* with respect to that ordering if  $T$  contains at most one bent edge and all primary edges and half-edges of  $T$  come before secondary edges and half-edges of  $T$  under the ordering.

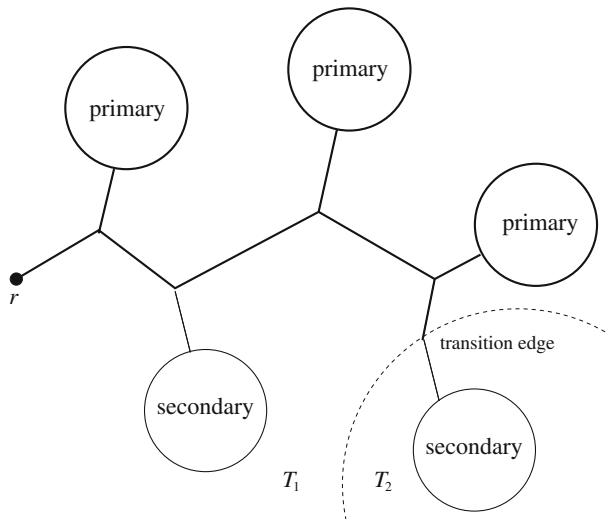
In a canonical tree there therefore exists a so-called *transition edge* (which is possibly a bent edge), such that all edges that appear before the transition edge wrt. the given ordering are *primary* and all edges appear after the transition edge are *secondary*. Hence a canonical tree has at most one bent edge.

The proof of the following theorem follows immediately from the fact that we can use zero-shifts to move primary/secondary components between edges in  $T$ . (A detailed proof is given in [5], and it works without modifications for the more general fixed orientation case.)

**Theorem 6.3** Let a set terminals  $N$  and a full Steiner topology  $\mathcal{T}$  for that set of terminals be given. Suppose there exists a full and fulsome SMT for  $N$  with topology  $\mathcal{T}$ . Then for any ordering of the edges of  $\mathcal{T}$  there exists an SMT for  $N$  (with topology  $\mathcal{T}$ ) which is canonical with respect to that ordering.

Depending on the chosen ordering, various canonical forms can be obtained [5]. From an algorithmic point of view, the *depth-first ordering* is the most important one: Root  $\mathcal{T}$  at some terminal  $r$  and order the edges as they appear in a depth-first traversal of the tree from  $r$ . This results in a canonical form that is illustrated in Fig. 13. If we divide such a canonical tree  $T$  into two subtrees  $T_1$  and  $T_2$  by deleting the transition edge, the tree  $T_2$  that does not contain the root is secondary (i.e., all edges in  $T_2$  are secondary edges). Furthermore, the path  $p$  from

**Fig. 13** Illustration of a tree  $T$  with the depth-first order canonical form. Primary edges are drawn as bold edges, and the remaining edges are secondary edges





$r$  to the transition edge is primary, while the subtrees that are attached to  $p$  are each either primary or secondary subtrees. These properties are used by the linear-time algorithm for constructing a SMT for a given topology in Sect. 7.

By choosing another ordering of the edges, we obtain the following interesting theorem [5]:

**Theorem 6.4** *Let  $T$  be a full and fulsome SMT for terminal set  $N$  with topology  $\mathcal{T}$ . Then there exists an SMT  $T'$  for terminal set  $N$  with topology  $\mathcal{T}$  that uses at most four legal orientations.*

*Proof* Order the edges of  $T$  by their *colour*; for example, the red edges may come first in the ordering followed by the green edges and then the blue edges. Consider a canonical SMT  $T'$  that comes from this ordering (by Theorem 6.3 such a tree exists). Then it follows from the canonical form that the edges of any given colour, not the same as the colour of the transition edge, are either all primary or all secondary. Hence  $T'$  uses at most four legal orientations.  $\square$

Finally, we note that the well-known Hwang canonical form for *rectilinear* trees [15,35] is a *special case* of the depth-first ordering canonical form. The first observation is that by Lemma 6.5, in a fulsome and canonical rectilinear SMT the topology of a full component must be a *chain* in which every Steiner point is connected directly to at least one terminal. The Hwang topology form is now obtained by rooting the topology in one end of the chain, and constructing the depth-first ordering canonical form. The fact that there can be at most two terminals in the subtree below the bent edge follows from the fullness of the tree.

## 7 Linear time algorithm for a given topology

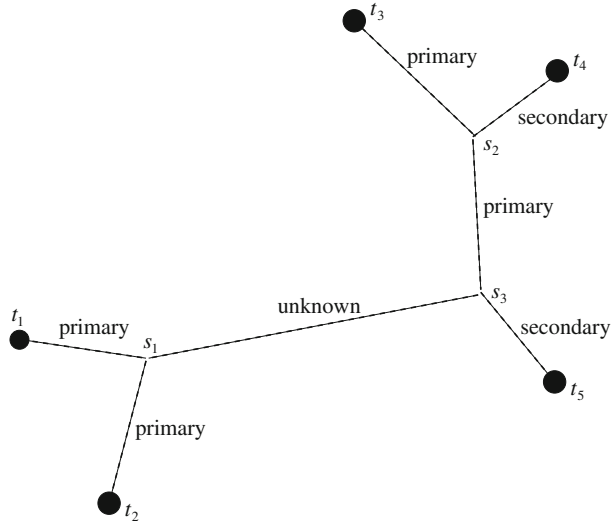
By exploiting the concept of canonical forms, in this section we present an  $O(\sigma n)$  time algorithm to compute a full and fulsome SMT  $T$  for a given full Steiner topology  $\mathcal{T}$  with  $n$  terminals (or show that no such tree exists). Our algorithm is a generalisation of the algorithm by Brazil et al. [5] which works for uniform orientation metrics with  $\lambda > 3$  given orientations. In the following we denote this as the *DFS-canonical algorithm*, since it is based on the canonical form that comes from a depth-first traversal of the topology  $\mathcal{T}$ . Here we first present the main algorithmic idea in Sect. 7.1. Then we show that the so-called merging operation can be performed in constant time (Sect. 7.2), and in Sect. 7.3 we briefly present the DFS-canonical algorithm and argue that it also works for the general fixed orientation metric.

### 7.1 Labelling and merging algorithm

Consider the topology  $\mathcal{T}$  shown in Fig. 14. This topology has five terminals  $t_1, \dots, t_5$ . We know from Sect. 5 that there exists a direction set  $\mathcal{U}$  that is used by every Steiner point in a full SMT  $T$  with topology  $\mathcal{T}$  (assuming that such a tree  $T$  exists). Assume furthermore that the primary/secondary labelling of the edges in  $T$  is as in Fig. 14. Hence the edge connecting, e.g., Steiner point  $s_1$  to terminal  $t_1$  is a straight edge with a primary direction from  $\mathcal{U}$ ; however, the colour of edge  $(s_1, t_1)$  is not known. Steiner point  $s_1$  is connected by straight and labelled edges to two nodes (here terminals) with *known* positions. As will be shown in Sect. 7.2, then in *constant time* we can either compute the *unique* location of  $s_1$ —or decide that  $s_1$  cannot exist with the given direction set and primary/secondary labelling.

The operation of computing the location of a node in  $\mathcal{T}$  given the locations of two of its neighbours is called a *merging* operation. If a direction set is given and all edges except one have been labelled primary or secondary (as in Fig. 14), then we can compute a tree under this

**Fig. 14** Labelling and bottom-up construction of SMT for a given topology spanning five terminals



labelling or decide that no such tree exists in  $O(n)$  time, where  $n$  is the number of terminals in  $T$ . We simply root  $T$  at the unlabelled (transition) edge and merge nodes bottom-up in  $T$ . In Fig. 14 we could, e.g., first compute  $s_1$  based on  $t_1$  and  $t_2$ , then  $s_2$  based on  $t_3$  and  $t_4$ , then  $s_3$  based on  $s_2$  and  $t_5$ , and finally construct the possibly bent transition edge based on the positions of the Steiner points  $s_1$  and  $s_3$ .

This is the main algorithmic idea employed in the linear-time algorithm. By using the canonical forms that come from ordering the edges in  $T$ , the number of possible labellings can be reduced from an exponential number—essentially two possible labels per edge in the tree—to a linear number of possible labellings (Sect. 7.3).

### 7.2 Constant-time merging operation

In order to facilitate the efficient bottom-up construction outlined in Sect. 7.1, we need a constant-time merging operation. The lemma below was originally proved only for uniform orientation metrics with  $\lambda > 3$  [5]. The generalization is rather complex and technical, and we refer to [7] for the full proof.

**Lemma 7.1** *Let  $T$  be a full SMT with full Steiner topology  $\mathcal{T}$ , and let  $s$  be a Steiner point in  $T$ . Assume that the locations in  $T$  of two of the neighbours of  $s$ ,  $u$  and  $v$ , are known; furthermore, assume that each edge  $(s, u)$  and  $(s, v)$  is straight and has been labelled primary or secondary. Finally, assume that we know the direction set for  $T$ . If Steiner point  $s$  exists in  $T$  and is not collinear with  $u$  and  $v$ , then its location is unique and can be computed in constant time; also, the colours of the edges  $(s, u)$  and  $(s, v)$ —and hence also the colour of the third edge incident with  $s$ —are unique.*

If  $s$ ,  $u$  and  $v$  are collinear, then again the the colours of the edges  $(s, u)$ ,  $(s, v)$  and the third edge incident with  $s$  are uniquely determined, but the position of  $s$  is not unique. This does not cause a problem in the bottom-up construction algorithm due to Lemma 6.5. As a consequence of this lemma, at most one merging operation in the bottom-up construction algorithm can result in a non-unique Steiner point  $s$ . In fact, this must then be the final merging operation where the third edge incident to  $s$  (in addition to  $(s, u)$  and  $(s, v)$ ) is the *transition*

edge, and the other end of this edge is a terminal  $t$ . This holds since the construction is moving in towards the transition edge. What we can do here is leave the position of the Steiner point undetermined and construct the transition edge by finding the shortest edge connecting  $t$  to the line segment  $uv$ . Clearly, this can also be done in constant time like an ordinary merging operation.

### 7.3 Linear-time DFS-canonical algorithm

In this section we show how the linear-time construction algorithm from [5] can be generalised to the fixed orientation metric. The algorithm uses the algorithmic idea outlined in Sect. 7.1, and the constant-time merging operation from Sect. 7.2. The algorithm is based on the depth-first traversal canonical form, and therefore denoted the DFS-canonical algorithm.

The DFS-canonical algorithm runs in  $O(n)$  time for a given direction set  $\mathcal{U}$ . In Sect. 4 we showed that the number of possible direction sets is  $\Theta(\sigma)$ , and that the direction sets furthermore can be determined in  $\Theta(\sigma)$  time. By iterating the  $O(n)$  time algorithm over all direction sets, we obtain a total running time of  $\Theta(\sigma) + O(\sigma n)$  which is  $O(\sigma n)$ .

The idea of the DFS-canonical algorithm (for a given topology  $\mathcal{T}$  and fixed direction set  $\mathcal{U}$ ) is that if there exists a full and fulsome SMT  $T$  with full Steiner topology  $\mathcal{T}$  and direction set  $\mathcal{U}$ , then  $T$  can be assumed to have the canonical form that is illustrated in Fig. 13. The algorithm consists of two depth-first traversals of  $\mathcal{T}$  from an arbitrary root terminal  $r$ :

1. Secondary subtrees are constructed bottom-up in  $\mathcal{T}$ , that is, subtrees where all edges are labelled as secondary edges. The merging operation is performed bottom-up for each Steiner node; if a Steiner point cannot be constructed, then none of the ancestors of the node (relative to  $r$ ) can be constructed either.
2. Primary subtrees are constructed bottom-up in  $\mathcal{T}$ . At the same time, each edge in  $\mathcal{T}$  is tried as potential transition edge. The idea is that it is possible iteratively to construct the endpoints of the transition in constant time per edge: The endpoint that is closest to the root  $r$  is constructed in the second traversal, while the other endpoint is constructed in the first traversal of  $\mathcal{T}$  (see Fig. 13).

Full implementation details are given by Brazil et al. [5]. Thus we have the following theorem:

**Theorem 7.1** *Given a full Steiner topology  $\mathcal{T}$  with  $n$  terminals and a fixed orientation metric with  $\sigma$  legal orientations, then in  $O(\sigma n)$  time we can either construct a full and fulsome SMT  $T$  with topology  $\mathcal{T}$ , or decide that no such tree exists.*

## 8 Conclusion

The main contribution of this paper is to show the existence of powerful canonical forms for SMTs for arbitrary (weighted) fixed orientation metrics. We also gave efficient and simple algorithms to compute all direction sets for a given fixed orientation metric, and to compute a SMT for a given full Steiner topology. Our results show that the two-phase exact algorithm for uniform orientation metrics [27] can easily be adapted to general fixed orientation metrics. This would most likely allow the efficient computation of large-scale SMTs for arbitrary fixed orientation metrics.

Future research will focus on generalising our results further to metrics given by piecewise differentiable unit circles—including resolving the conjecture that in any normed plane and

for any set of terminals there exists an SMT where the edges of each full component use at most *three* orientations (where the orientation of an edge is defined by the straight line between the endpoints of the edge).

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